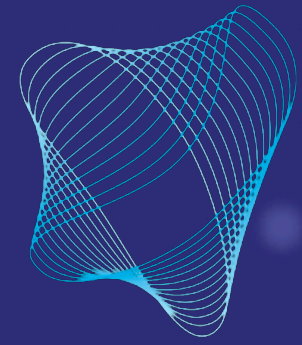




Center for
Quantum Networks
NSF Engineering Research Center

<https://cqn-erc.org/>

Modes, States, Transformations and Measurements

Instructor: Saikat Guha

Co-Director, NSF ERC Center for Quantum Networks,
Clark Distinguished Chair Professor of ECE,
University of Maryland College Park
Adjunct Professor of Optical Sciences, University of Arizona

Co-Instructor: Gabe Richardson

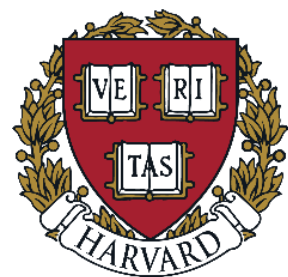
Graduate Research Associate,
University of Maryland College Park

This work is supported primarily by the Engineering Research Centers Program of the National Science Foundation. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect those of the National Science Foundation.

January 5, 2026

2026 CQN Winter School on Quantum Networks

Funded by National Science Foundation Grant #1941583



Outline

Part 1

A. Modes (Gabe)

B. Coherent State of Mode(s) (Saikat)

Break

Part 2

C. Quantum States of Mode(s) (Gabe)

D. State Discrimination (Saikat)



You will need a pen and paper to work out most of the exercises

Starr Pass Trail, Tucson AZ, April 2021
w/ Prof. Dirk Englund, MIT



Part A

Modes

Shapes in space and time

Definition of a ‘mode’

- An optical mode is the “shape” of a confined EM field in space, time, polarization
 - 4 independent attributes of the photon field, denoted by variables: (x, y, t, ν)
 - Recall: Time & Frequency are essentially the same variable (related by FT):
- Mode function: $\phi_\nu(\mathbf{r}, t); \mathbf{r} \in \mathcal{A}, t \in [0, T), \nu = 1, 2 \quad \mathbf{r} \equiv (x, y)$

$\nearrow$ Polarization $\nearrow$ Space $\nearrow$ Time

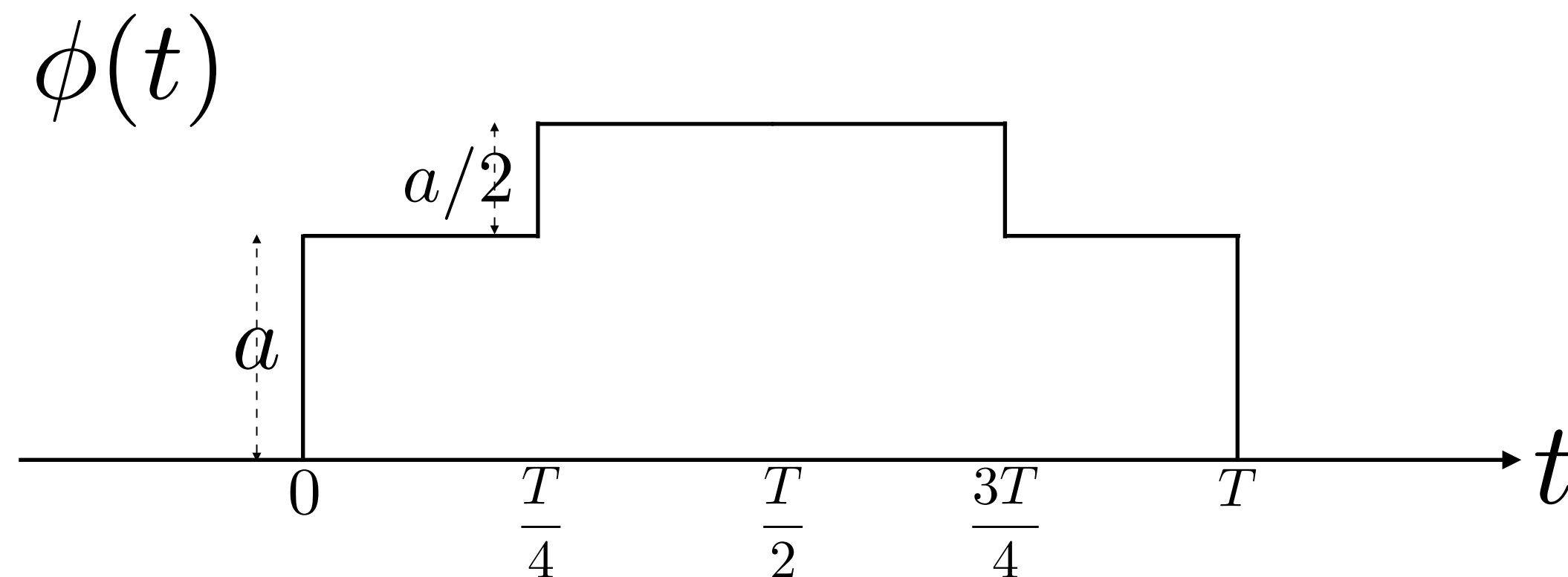
Polarization index = subscript

- We will always take a mode to be a unit-norm (complex-valued) function, i.e.,

$$\int_{\mathcal{A}} \int_0^T \phi_\nu(\mathbf{r}, t) \phi_\nu^*(\mathbf{r}, t) d\mathbf{r} dt = \int_{\mathcal{A}} \int_0^T |\phi_\nu(\mathbf{r}, t)|^2 d\mathbf{r} dt = 1$$

Exercise 1: Normalization of a mode function

- Consider the following mode function



- What is the value of a such that $\phi(t)$ is a properly normalized mode function?

A. $a = \frac{4}{5T}$

C. $a = \frac{8}{13T}$

B. $a = \sqrt{\frac{8}{11T}}$

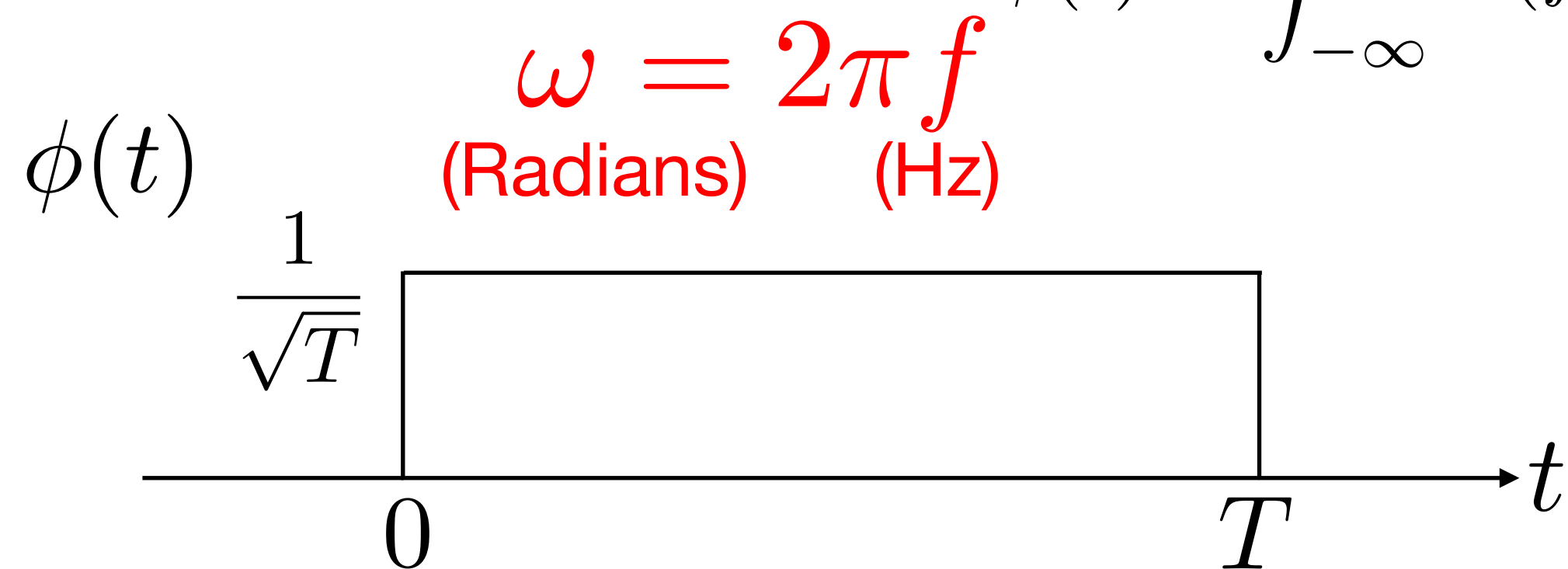
D. $a = \sqrt{\frac{8}{13T}}$ ←

Time and Frequency

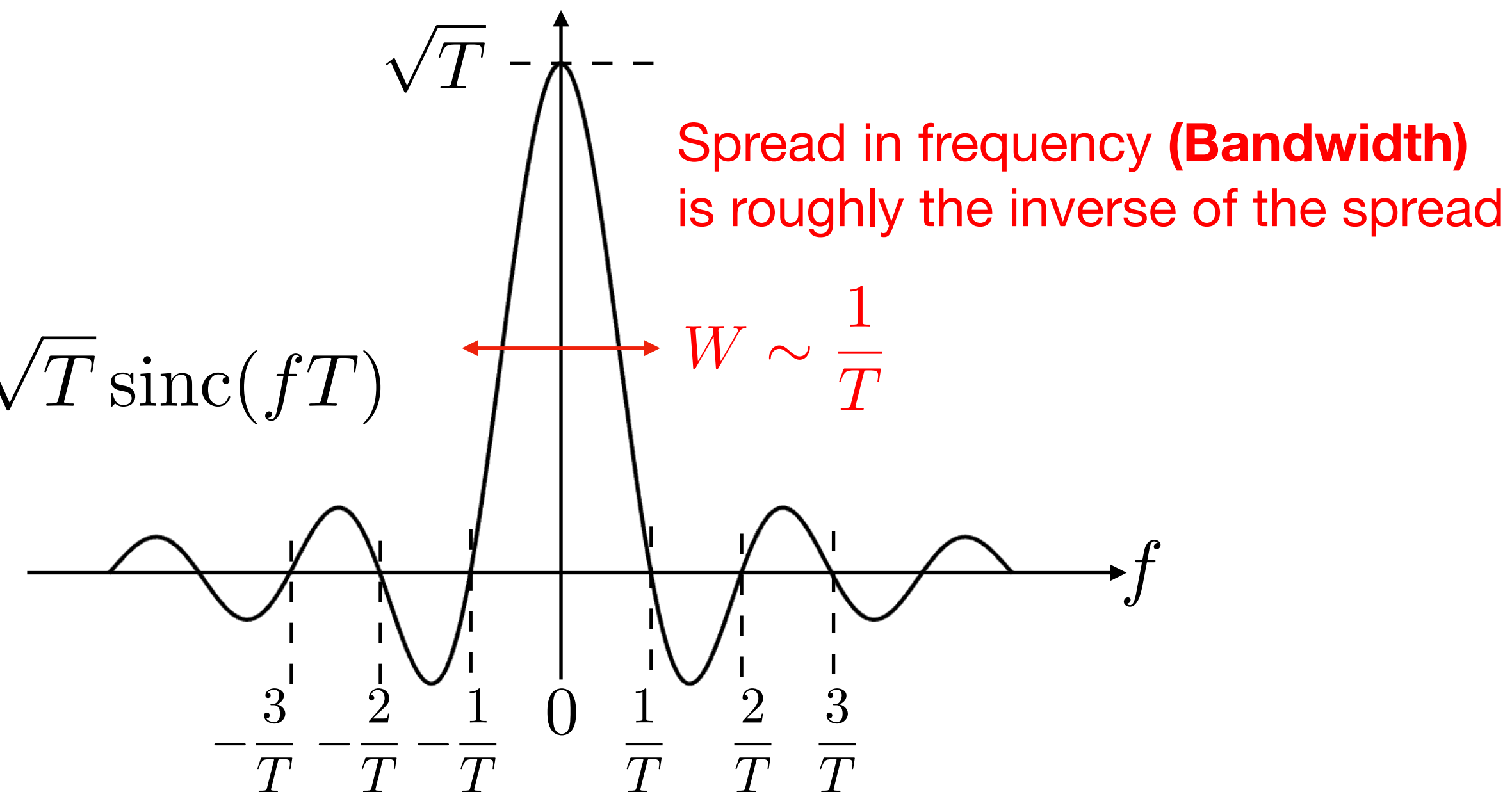
- Consider the “flat top” temporal mode, $\phi(t) = \begin{cases} \frac{1}{\sqrt{T}}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$

- Fourier transform (FT), $\Phi(f) = \int_{-\infty}^{\infty} \phi(t) e^{-2\pi i f t} dt$

- Inverse transform, $\phi(t) = \int_{-\infty}^{\infty} \Phi(f) e^{2\pi i f t} df$



$$\Phi(f) = \sqrt{T} \operatorname{sinc}(fT)$$



$$\phi(t) \leftrightarrow \Phi(f)$$

$$\phi(t) e^{2\pi i f_0 t} \leftrightarrow \Phi(f - f_0)$$

$$\phi(t - t_0) \leftrightarrow \Phi(f) e^{-2\pi i f t_0}$$

Parseval's theorem

$$\int_{-\infty}^{\infty} |\phi(t)|^2 dt = \int_{-\infty}^{\infty} |\Phi(f)|^2 df$$

Orthogonal modes, Time-bandwidth product

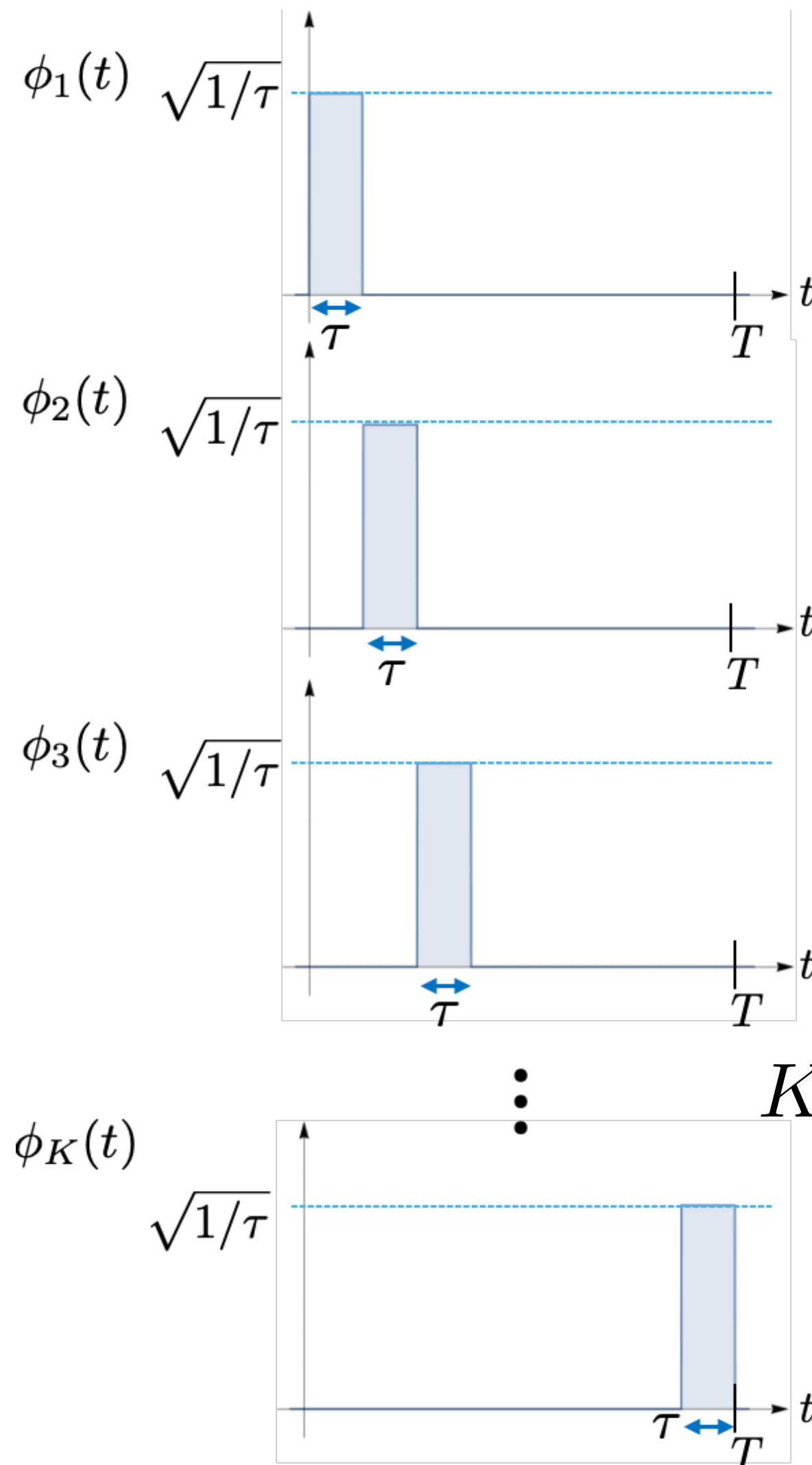
- Two modes $\phi_\nu(\mathbf{r}, t)$ and $\psi_\mu(\mathbf{r}, t)$ are orthogonal if, $\int_{\mathcal{A}} \int_0^T \phi_\nu(\mathbf{r}, t) \psi_\mu^*(\mathbf{r}, t) d\mathbf{r} dt = 0$
 - two modes will be orthogonal regardless of their spatial and temporal shapes, if $\nu \neq \mu$
 - If $\nu = \mu$, we will drop the polarization subscript
 - When we say two “temporal modes” $\phi_1(t)$ and $\phi_2(t)$ are orthogonal, we will implicitly assume the mode functions being referred to have the same spatial modes and same polarization
- How many orthogonal temporal modes can reside in a “time-bandwidth product”?
 - Orthonormal modes, $\phi_k(t)$, $k = 1, \dots, K$, Fourier transforms: $\Phi_k(f) = \int_{-\infty}^{\infty} \phi_k(t) e^{-2\pi i f t} dt$ $\phi_k(t) = \int_{-\infty}^{\infty} \Phi_k(f) e^{2\pi i f t} df$
We are using k to index orthogonal modes in a set (of same polarization). k is NOT the polarization index
 - Time limited (T sec): $\phi_k(t) = 0, t \notin [0, T)$, and (roughly) band-limited (W Hz): $\Phi_k(f) \approx 0, f - f_0 > \left| \frac{W}{2} \right|$
 - They are orthogonal: $\int_0^T \phi_k(t) \phi_l^*(t) dt = \delta_{kl}$, i.e., $\int_{-\infty}^{\infty} \Phi_k(f) \Phi_l^*(f) df = \delta_{kl}$
 - Maximum number of modes that can satisfy the above conditions, $K \approx WT$



Filling up the Time-Bandwidth product



Pixel modes: Each mode fills up the same BW
W, but different time extents within [0, T)

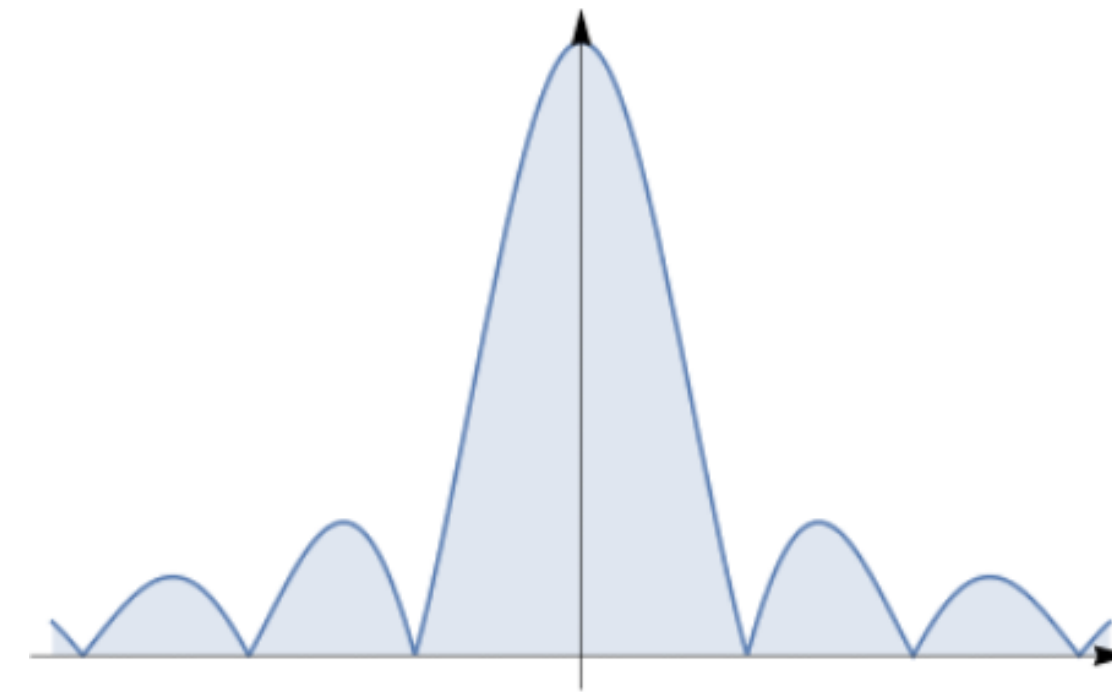


$$W \sim \frac{1}{\tau}$$

Fourier Transform
→

$$\Phi_k(f) = \sqrt{\tau} \text{sinc}(f\tau) \times e^{-2\pi i f(k-1)\tau}$$

$$\text{abs}[\Phi_k(f)] = \sqrt{\tau} |\text{sinc}(f\tau)|$$



Freq. description of every pixel mode
is the same (in absolute value)

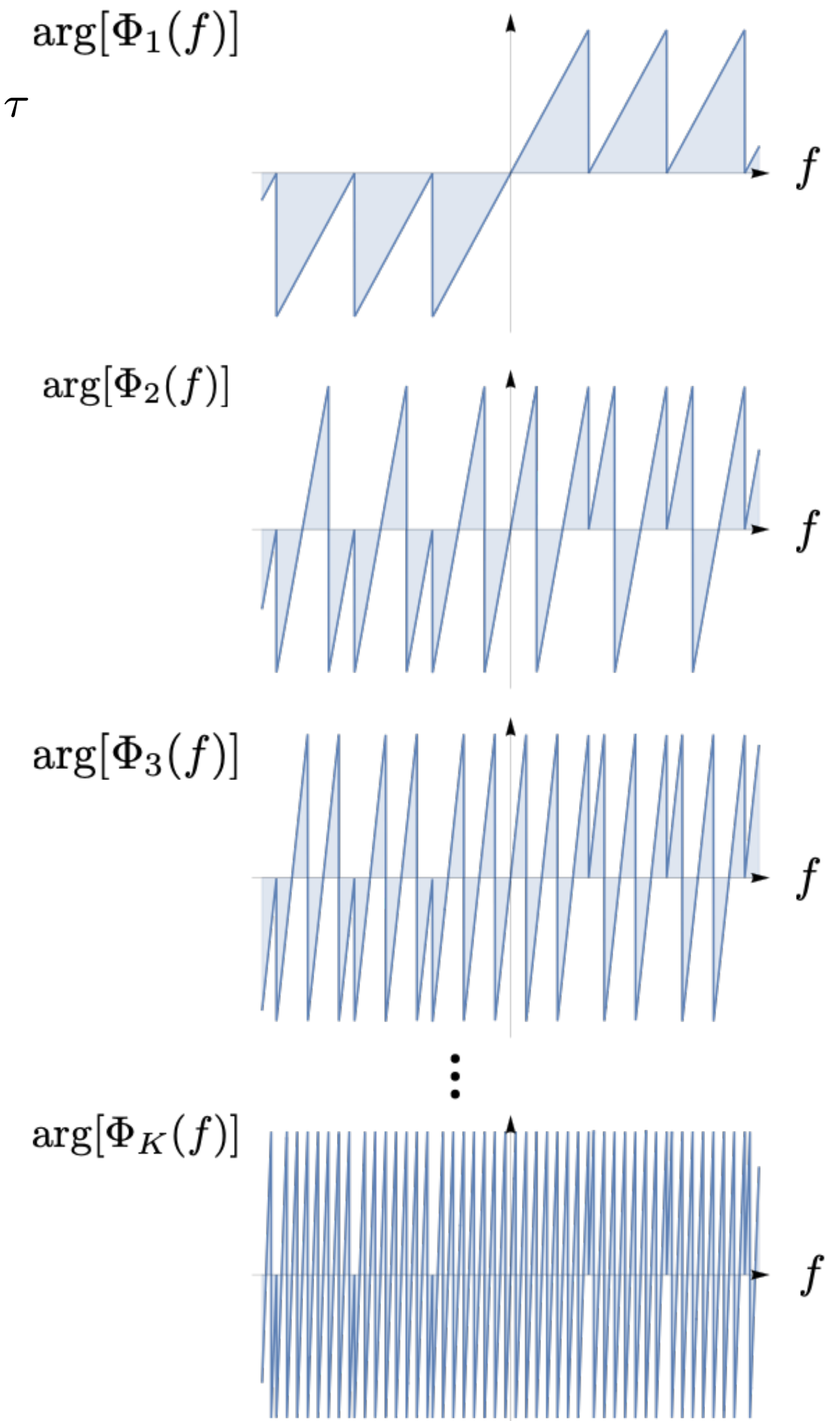
$$\text{abs}[\Phi_k(f)]$$

$$\forall k \in \{1, 2, \dots, K\}$$

Recall that shifting in time $\equiv$ phase
shift of Fourier transform

$$\phi(t - t_0) \leftrightarrow \Phi(f) e^{-2\pi i f t_0}$$

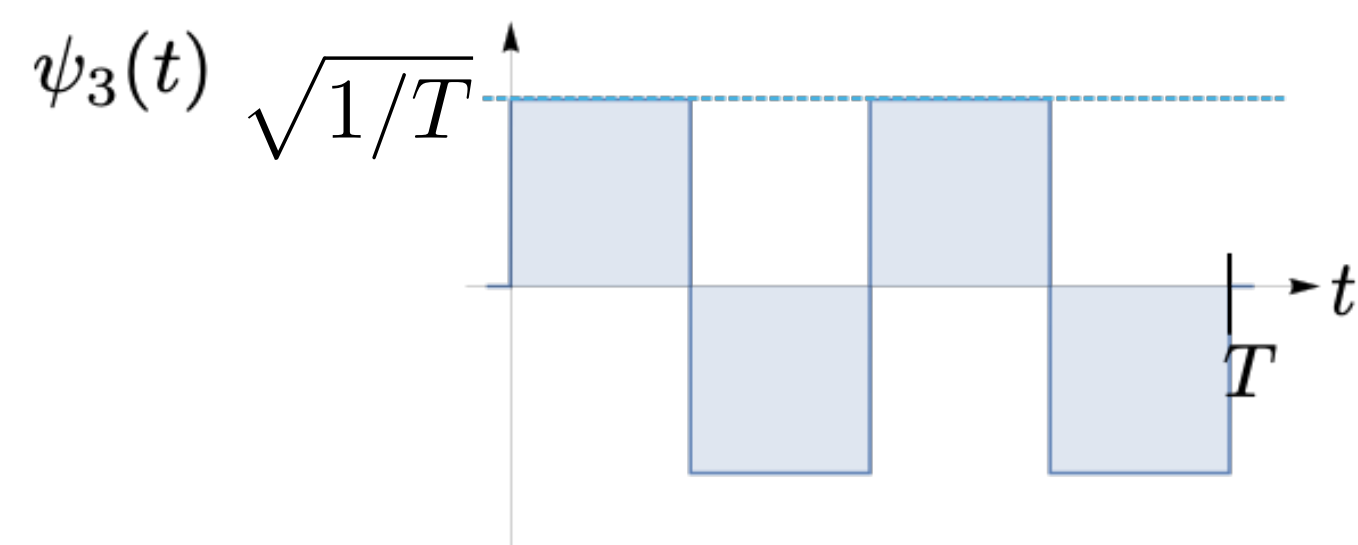
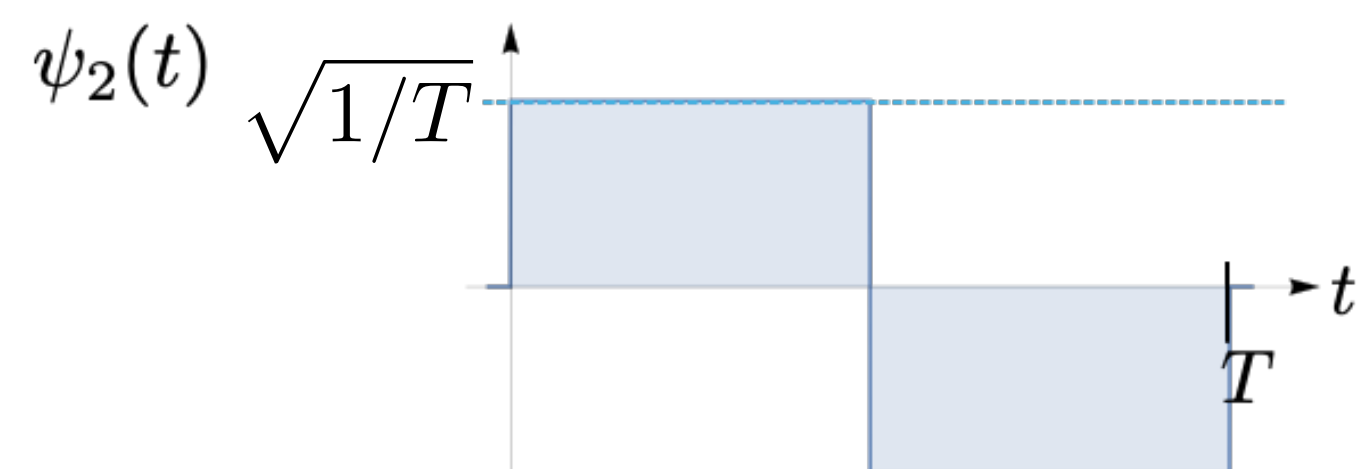
$$\arg[\Phi_k(f)] = 2\pi f(k-1)\tau$$



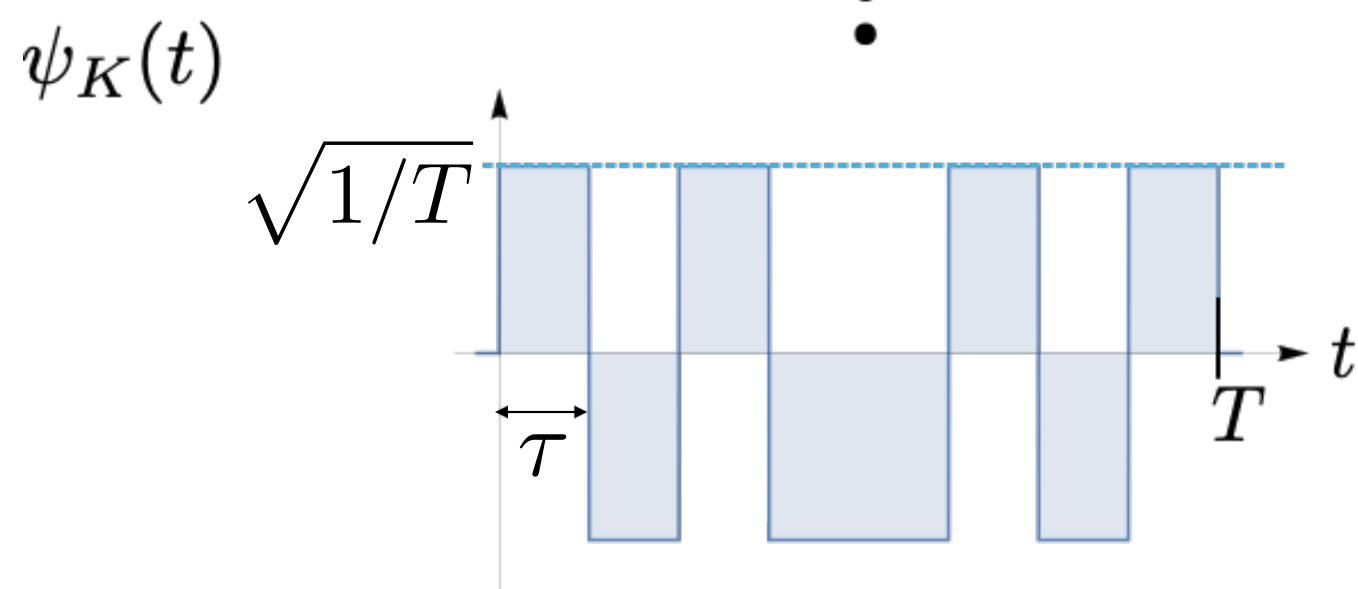
The phase fluctuations are different
for each mode

Filling up the Time-Bandwidth product

Walsh-Hadamard modes: Each mode fills up the same time $[0, T)$, but different freq extents within $[-W/2, W/2]$

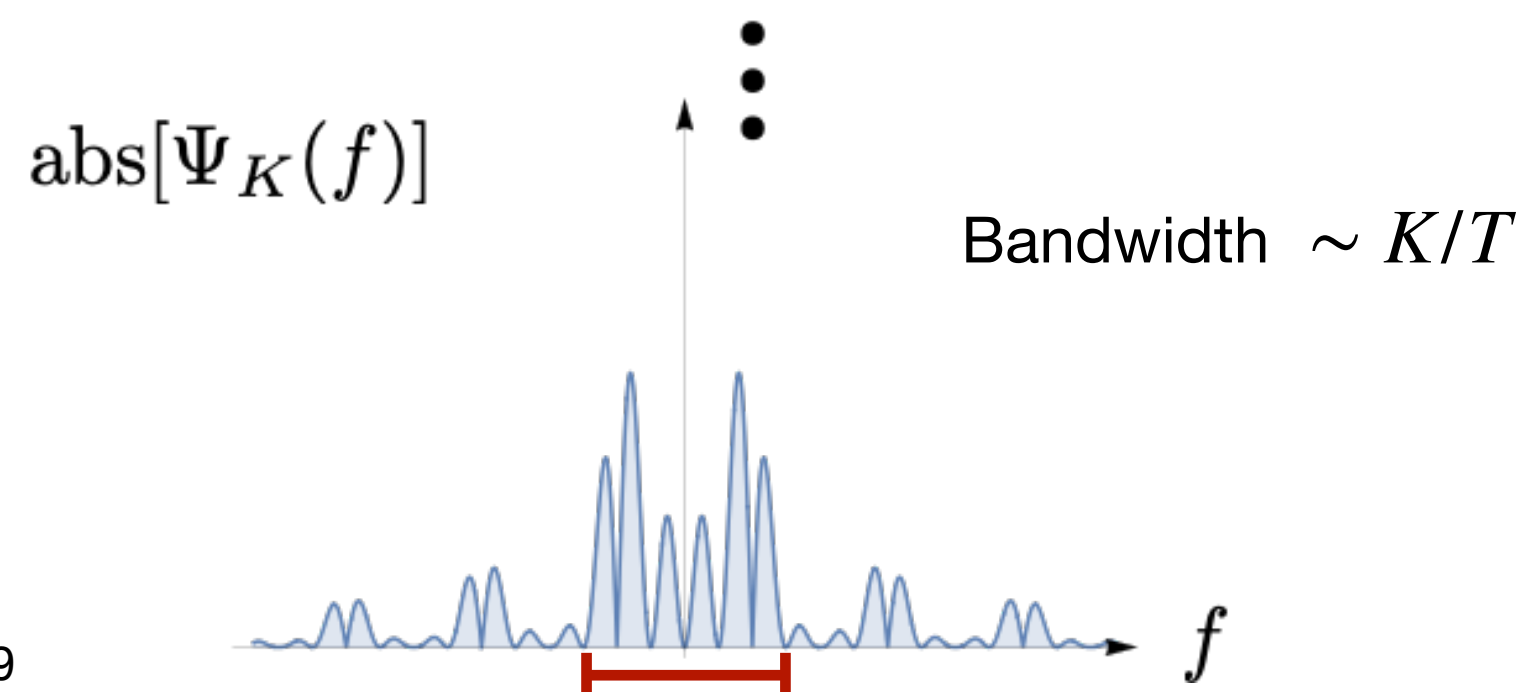
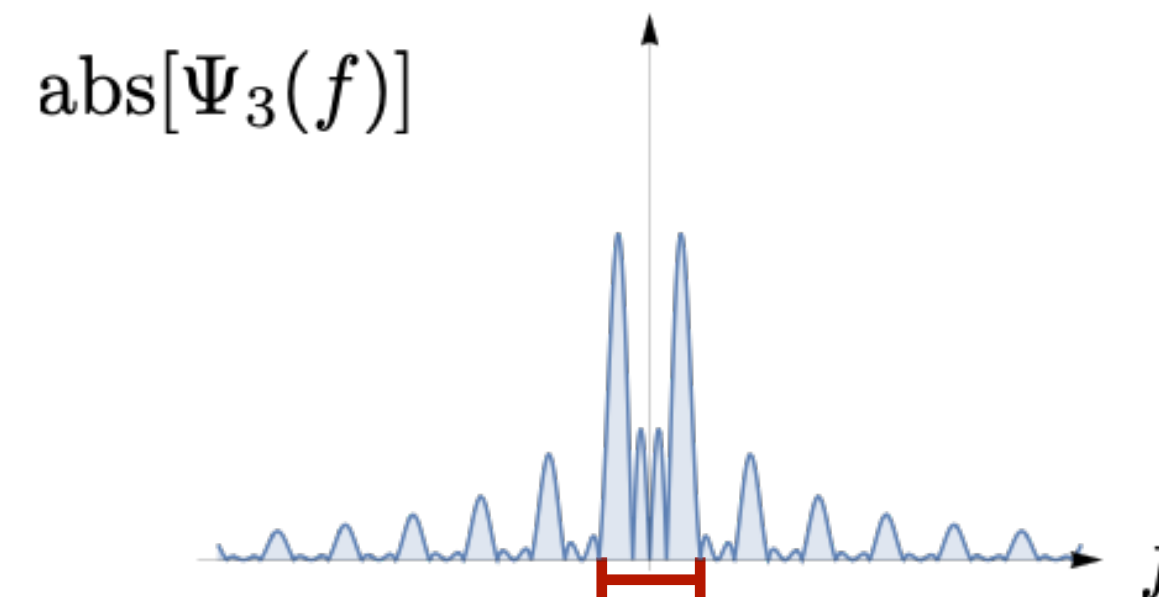
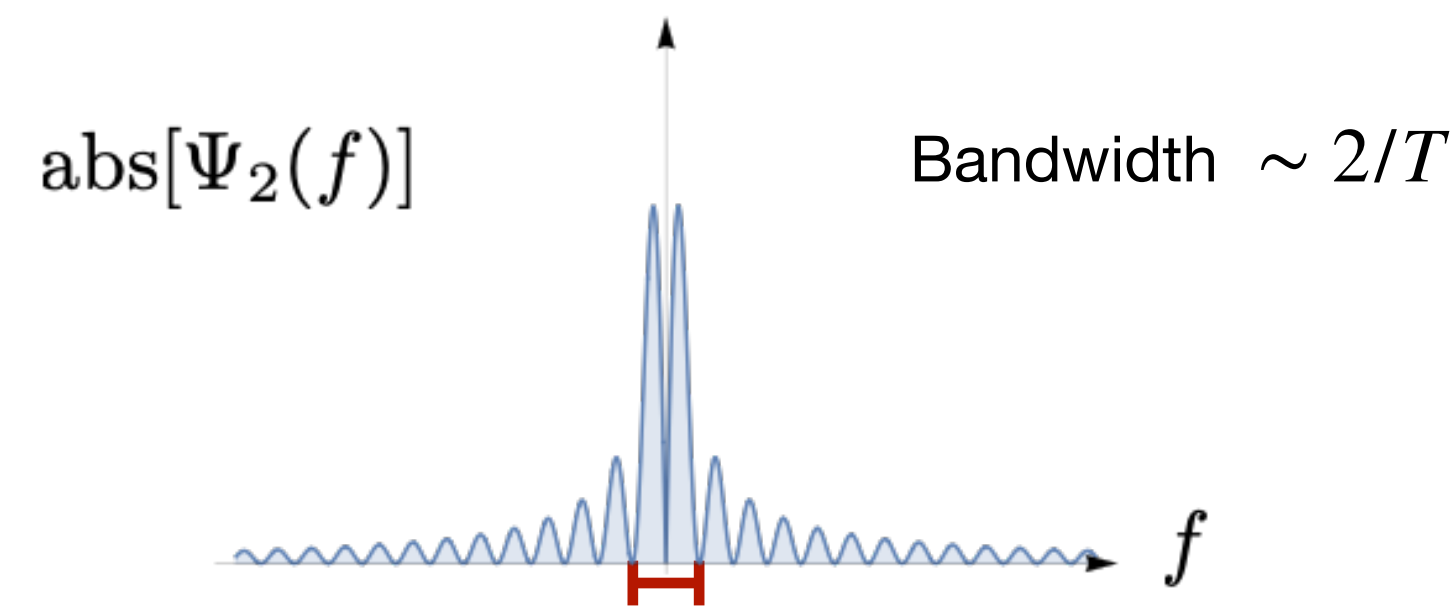
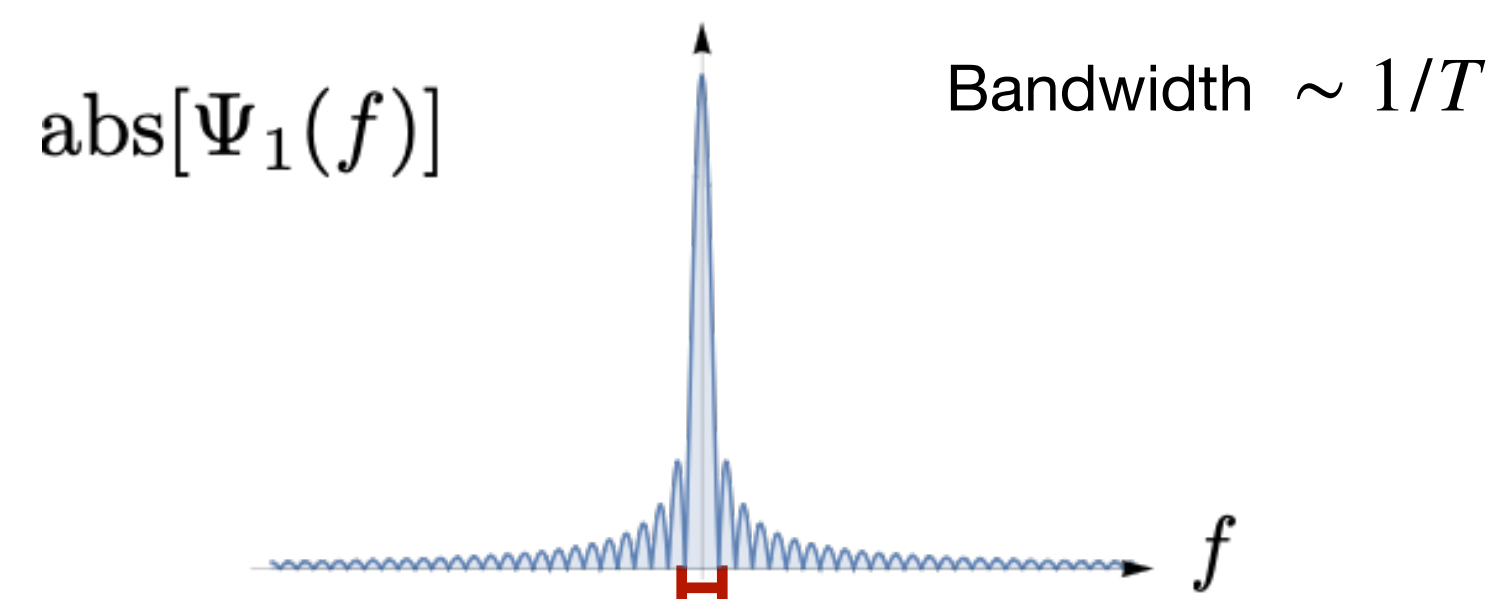


⋮



$$\tau = \frac{T}{K}$$

Fourier Transform



Note that each function has a different frequency bandwidth (shown by red marker)

Great Falls, MD
October 5, 2024



Part B

Coherent State of Mode(s)

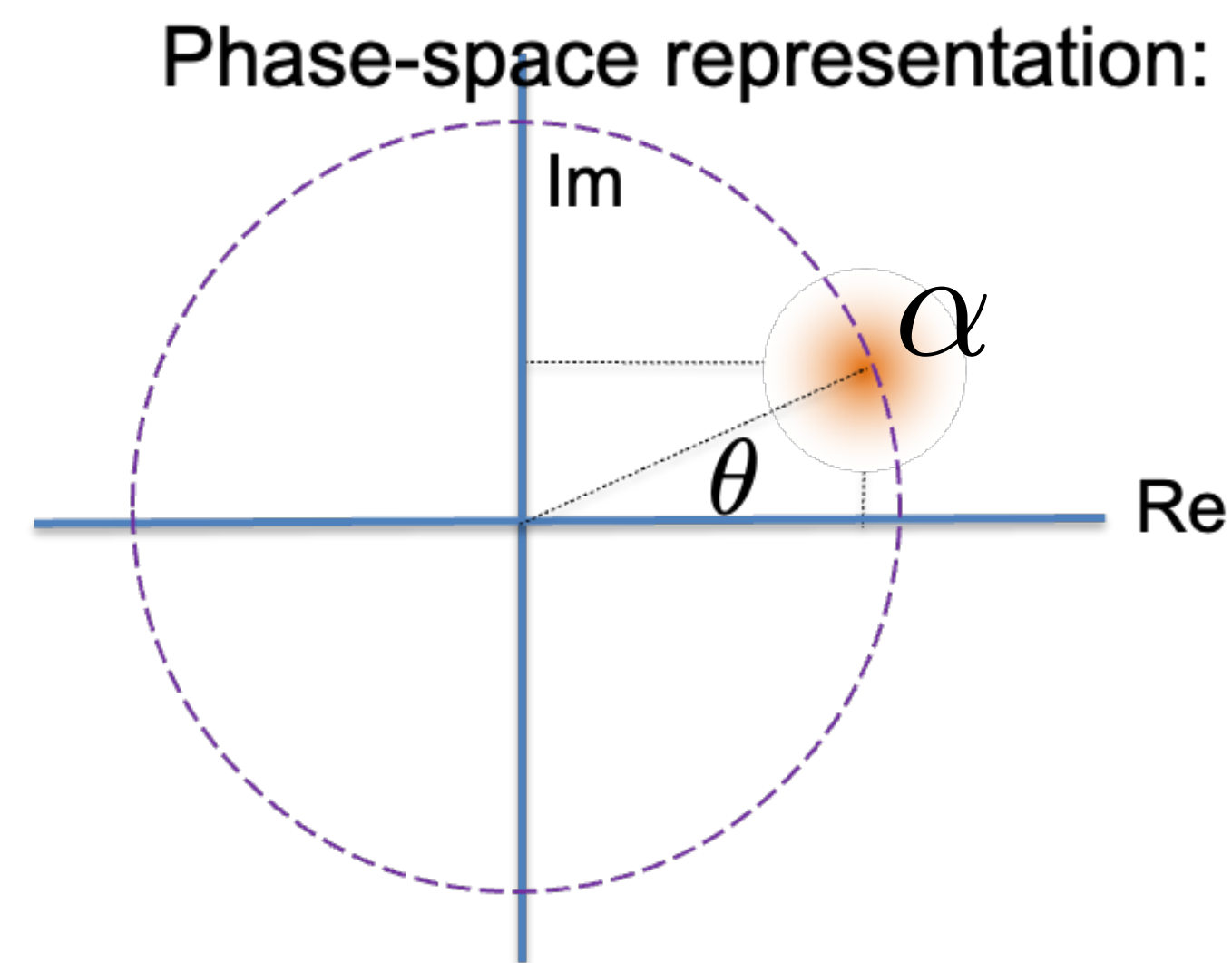
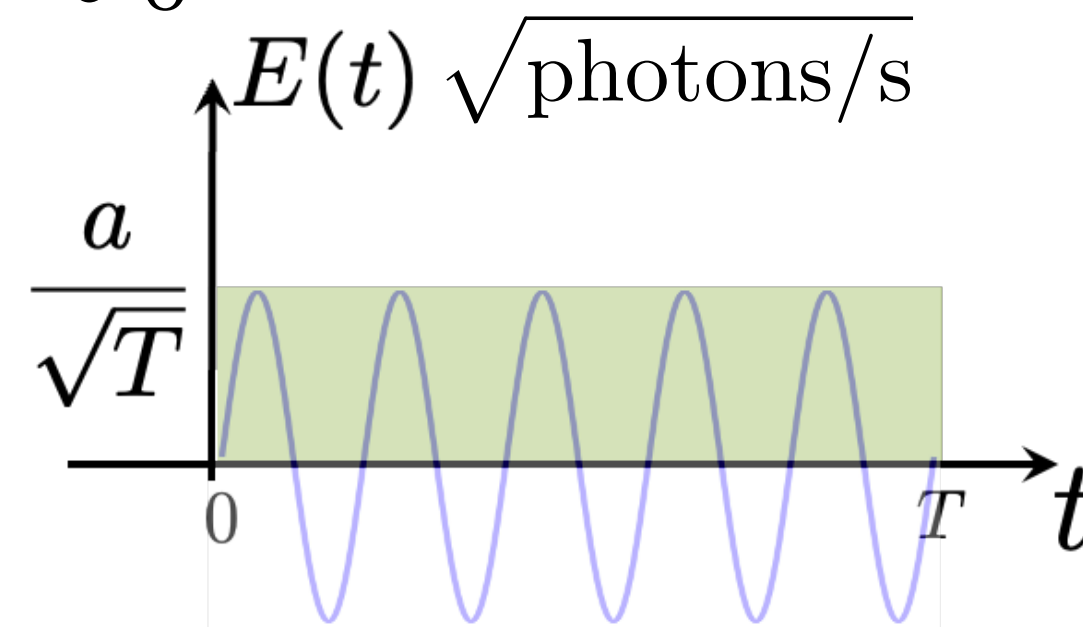
Classical Electromagnetic (EM) Field

Coherent state of a mode

Semiclassical: Pulse of laser light

Quantum: coherent state of a mode

- Consider the “flat top” temporal mode: $\phi(t) = \begin{cases} \frac{1}{\sqrt{T}}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$
- Electric field of an ideal laser-light pulse: $E(t) = a\phi(t)e^{i(2\pi f_0 t + \theta)} \sqrt{\text{photons/sec}}$
- Mean photon number in the pulse, $N = \int_0^T |E(t)|^2 dt = a^2$
- Quantum description:
 - Coherent state $|\alpha\rangle$ of the mode $\phi(t)$,
 - with $\alpha = \sqrt{N} e^{i\theta} = a e^{i\theta}$

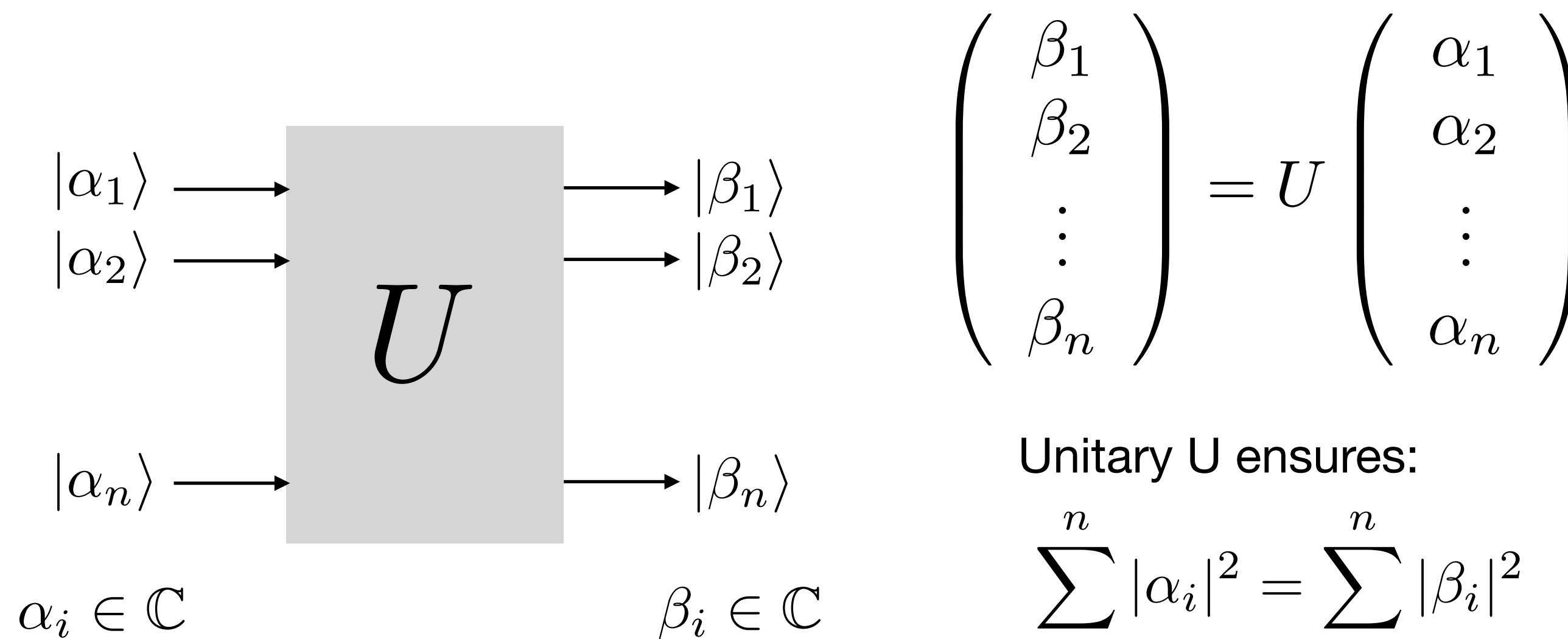


Coherent state of a set of (orthogonal) modes

- Consider two orthonormal modes, $\phi_1(t)$ and $\phi_2(t)$
- Their *superposition*, $\phi(t) = c_1\phi_1(t) + c_2\phi_2(t)$ is a mode, given $|c_1|^2 + |c_2|^2 = 1$
- Consider the field, $E(t) = a\phi(t)e^{i(2\pi f_0 t + \theta)} = (ac_1e^{i\theta}\phi_1(t) + ac_2e^{i\theta}\phi_2(t))e^{i2\pi f_0 t}$
- Quantum description:
 - Coherent state $|\alpha\rangle$; $\alpha = ae^{i\theta}$, of mode $\phi(t)$, OR
 - Coherent state $|\alpha_1\rangle_{\phi_1} \otimes |\alpha_2\rangle_{\phi_2}$, $\alpha_j = ac_je^{i\theta}$, $j = 1, 2$, of modes $\phi_1(t)$ and $\phi_2(t)$
 $\equiv |\alpha_1\rangle|\alpha_2\rangle$ We will drop the mode label on the kets when context is clear, and will drop the tensor product sign
- Most general description: quantum state of a collection of orthonormal modes
 - When we write a ket (to describe a quantum state), we have a mode (or collection of orthogonal modes) in mind
 - Coherent state is single mode: can always find a mode s.t. the state of the field is a single-mode coherent state

Coherent state of a set of (orthogonal) modes

- Coherent state, when expressed in any orthonormal mode basis, $\{\phi_1(t), \dots, \phi_n(t)\}$ is a “product state” of the form $|\alpha_1\rangle_{\phi_1(t)} |\alpha_2\rangle_{\phi_2(t)} \dots |\alpha_n\rangle_{\phi_n(t)}$
- Transforming modal basis: a unitary (reversible) operation



$\alpha_i \in \mathbb{C}$

$\beta_i \in \mathbb{C}$

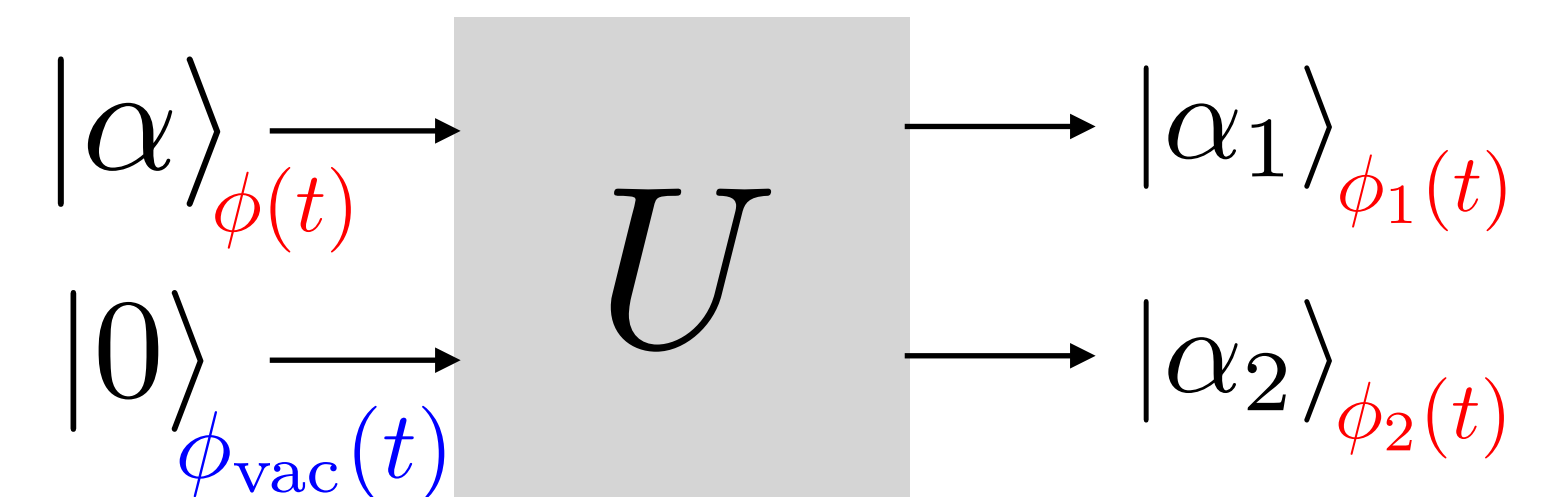
Complex valued
unitary matrix

$$U^\dagger U = \mathcal{I}$$

For any $(\alpha_1, \dots, \alpha_n)$,
there exists a U , s.t.,

$$\beta_1 \neq 0, \beta_2 = \dots = \beta_n = 0$$

Example of change of basis from
previous slide can be seen as a
unitary transformation



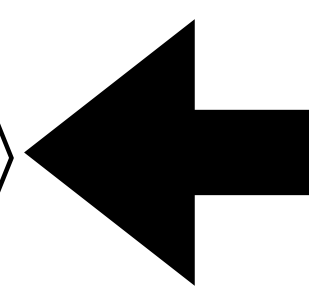
“Beam Splitter” unitary

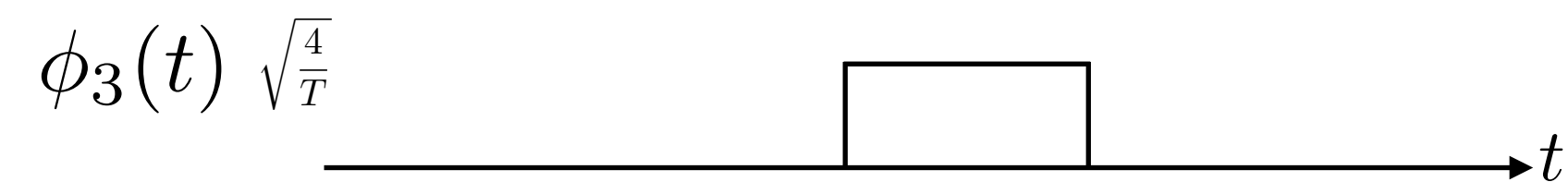
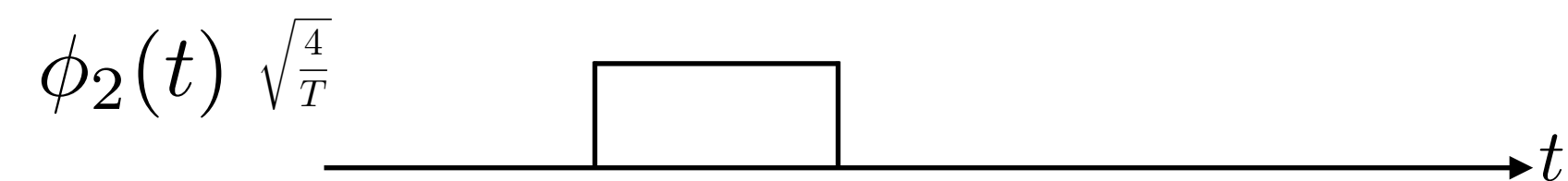
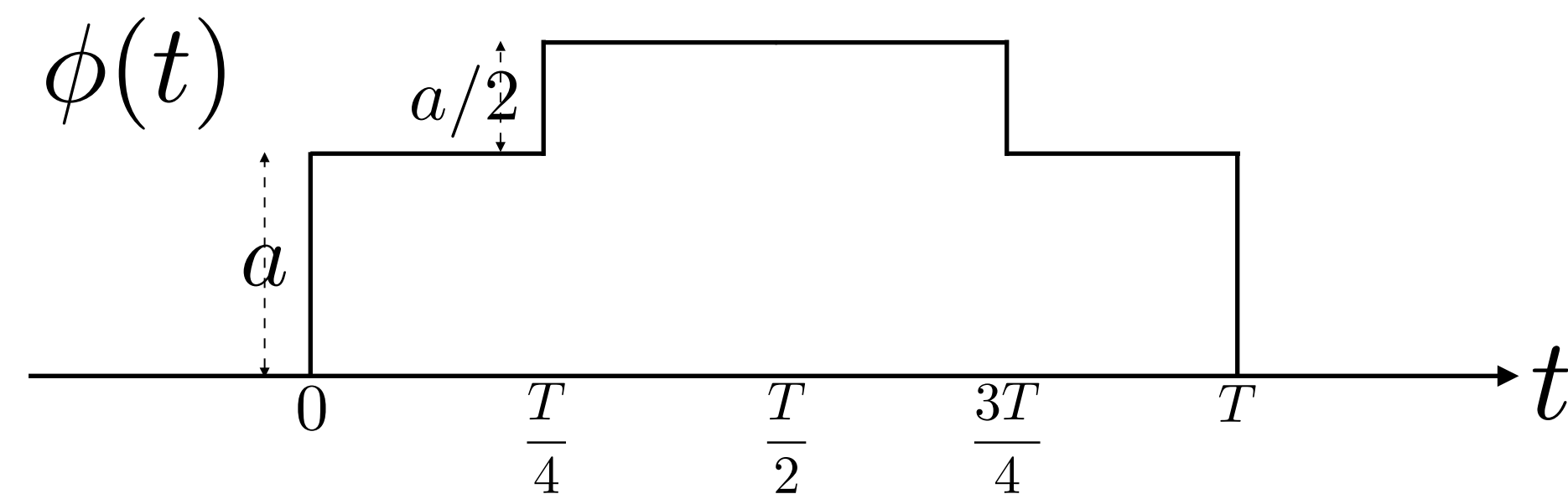
Exercise 2A: Changing the mode basis

- Represent coherent state $|\alpha\rangle$ of the shown mode $\phi(t)$ in:

$$a = \sqrt{\frac{8}{13T}}$$

- The Pixel Basis:

- A. $\left|\sqrt{\frac{2}{13}}\alpha\right\rangle \left|\frac{1}{2}\sqrt{\frac{2}{13}}\alpha\right\rangle \left|\frac{1}{2}\sqrt{\frac{2}{13}}\alpha\right\rangle \left|\sqrt{\frac{2}{13}}\alpha\right\rangle$
- B. $\left|\sqrt{\frac{2}{13}}\alpha\right\rangle \left|\sqrt{\frac{9}{26}}\alpha\right\rangle \left|\sqrt{\frac{9}{26}}\alpha\right\rangle \left|\sqrt{\frac{2}{13}}\alpha\right\rangle$ 
- C. $\left|\sqrt{\frac{2}{13}}\alpha\right\rangle \left|\sqrt{\frac{9}{13}}\alpha\right\rangle \left|\sqrt{\frac{9}{13}}\alpha\right\rangle \left|\sqrt{\frac{2}{13}}\alpha\right\rangle$
- D. $\left|\sqrt{\frac{2}{13}}\alpha\right\rangle \left|\frac{1}{2}\sqrt{\frac{5}{13}}\alpha\right\rangle \left|\frac{1}{2}\sqrt{\frac{5}{13}}\alpha\right\rangle \left|\sqrt{\frac{2}{13}}\alpha\right\rangle$



Hint:

$$\phi(t) = \sqrt{\frac{T}{4}} \left[a\phi_1(t) + \frac{3a}{2}\phi_2(t) + \frac{3a}{2}\phi_3(t) + a\phi_4(t) \right]$$

Exercise 2B: Changing the mode basis

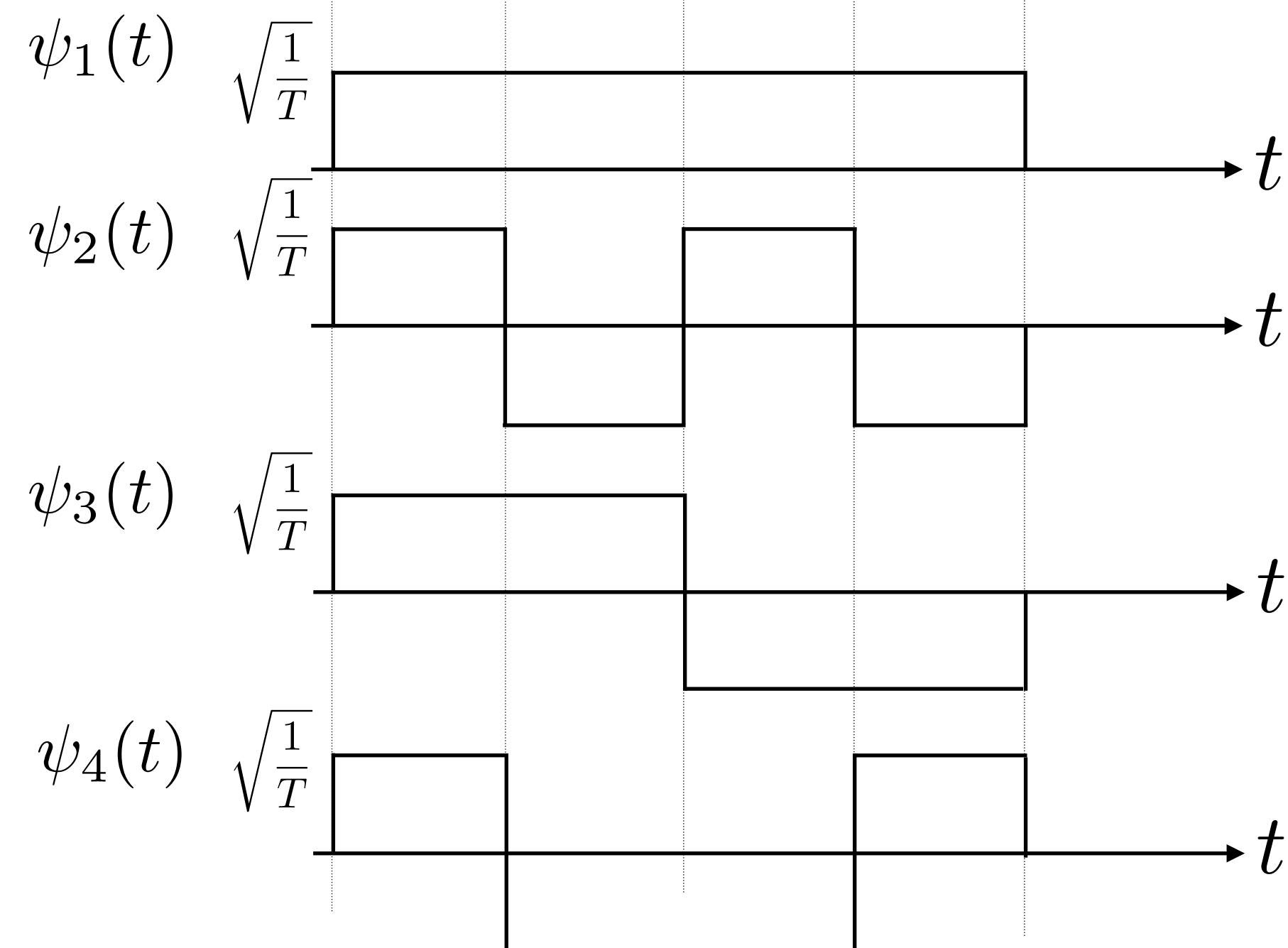
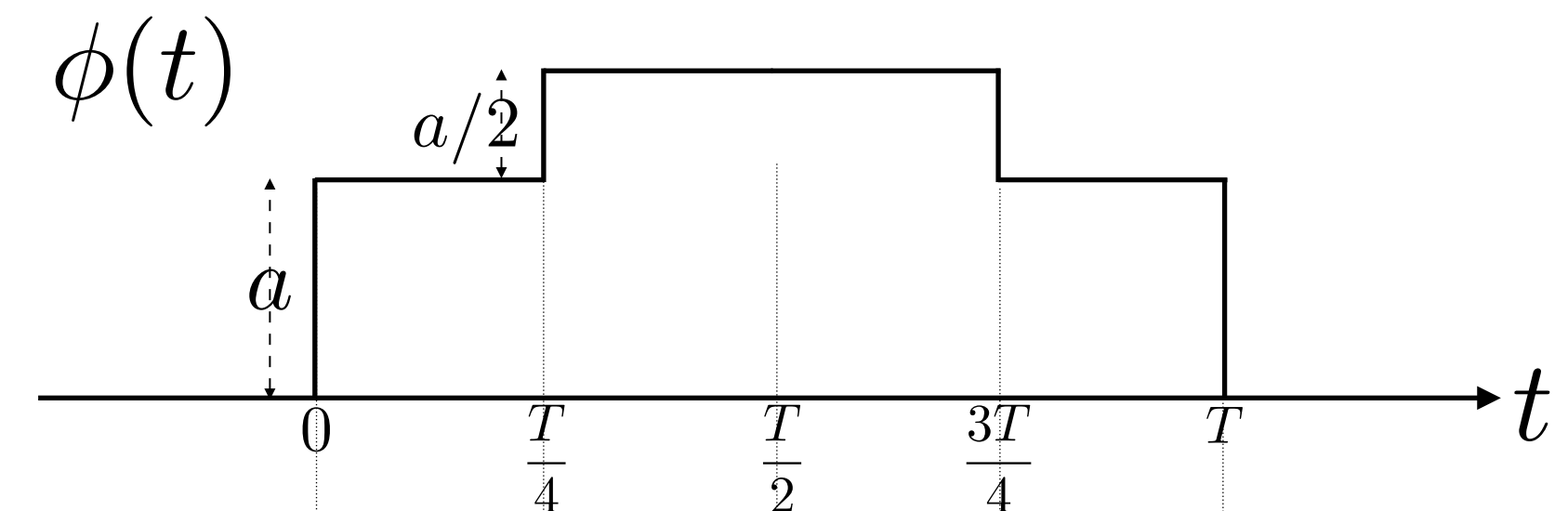
- Represent coherent state $|\alpha\rangle$ of the shown mode $\phi(t)$ in,

Hint:

$$\begin{aligned}\phi_1(t) &= \frac{1}{2}[\psi_1(t) + \psi_2(t) + \psi_3(t) + \psi_4(t)] \\ \phi_2(t) &= \frac{1}{2}[\psi_1(t) - \psi_2(t) + \psi_3(t) - \psi_4(t)] \\ \phi_3(t) &= \frac{1}{2}[\psi_1(t) + \psi_2(t) - \psi_3(t) - \psi_4(t)] \\ \phi_4(t) &= \frac{1}{2}[\psi_1(t) - \psi_2(t) - \psi_3(t) + \psi_4(t)]\end{aligned}$$

$$\phi(t) = \sqrt{\frac{T}{4}} \left[a\phi_1(t) + \frac{3a}{2}\phi_2(t) + \frac{3a}{2}\phi_3(t) + a\phi_4(t) \right]$$

$$\phi(t) = \sqrt{\frac{25}{26}}\psi_1(t) - \sqrt{\frac{1}{26}}\psi_4(t)$$



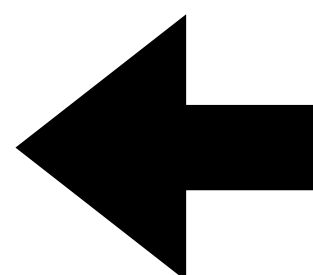
- The Walsh-Hadamard basis:

- A. $\left| \sqrt{\frac{25}{26}}\alpha \right\rangle |0\rangle |0\rangle \left| \sqrt{\frac{1}{26}}\alpha \right\rangle$

- B. $\left| -\sqrt{\frac{25}{26}}\alpha \right\rangle |0\rangle |0\rangle \left| \sqrt{\frac{1}{26}}\alpha \right\rangle$

- C. $\left| \sqrt{\frac{25}{26}}\alpha \right\rangle |0\rangle |0\rangle \left| -\sqrt{\frac{1}{26}}\alpha \right\rangle$

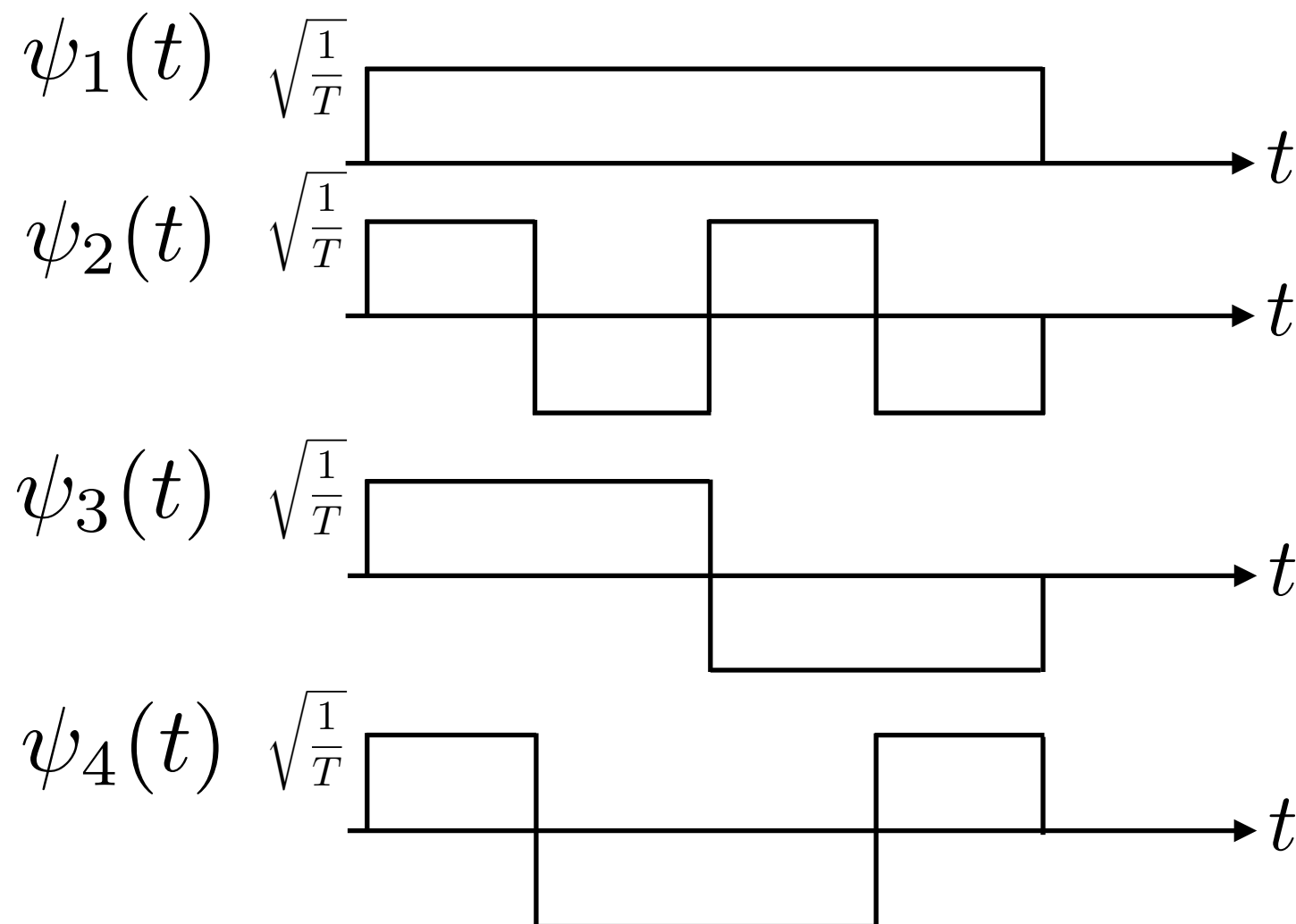
- D. $\left| -\sqrt{\frac{25}{26}}\alpha \right\rangle |0\rangle |0\rangle \left| -\sqrt{\frac{1}{26}}\alpha \right\rangle$



Mode sorting

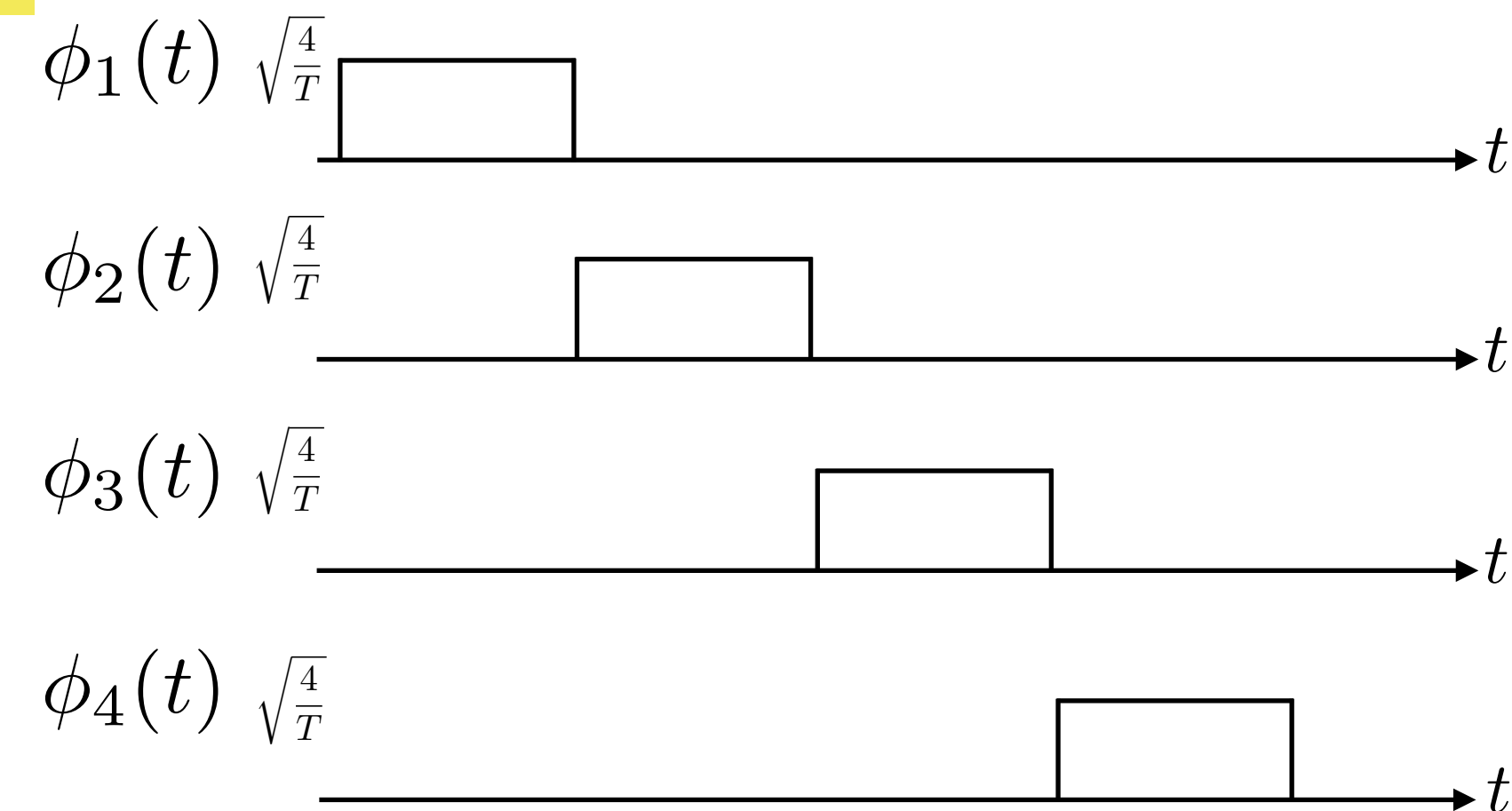
- Unitary mode transformation from a chosen mode basis to the pixel basis:

$$E_{\text{in}}(t) = a_1\psi_1(t) + a_2\psi_2(t) + a_3\psi_3(t) + a_4\psi_4(t)$$



Mode
Sorter

$$E_{\text{out}}(t) = a_1\phi_1(t) + a_2\phi_2(t) + a_3\phi_3(t) + a_4\phi_4(t)$$



- Examples of hardware for sorting polarization, spatial, and temporal modes
 - Hologram, Prism, Quantum Pulse Gate (QPG), Multi-plane light conversion (MPLC), Engineered “meta material” (numerically computed refractive-index distribution)
 - We will study the MPLC in a little more detail

Multi Plane Light Conversion (MPLC)

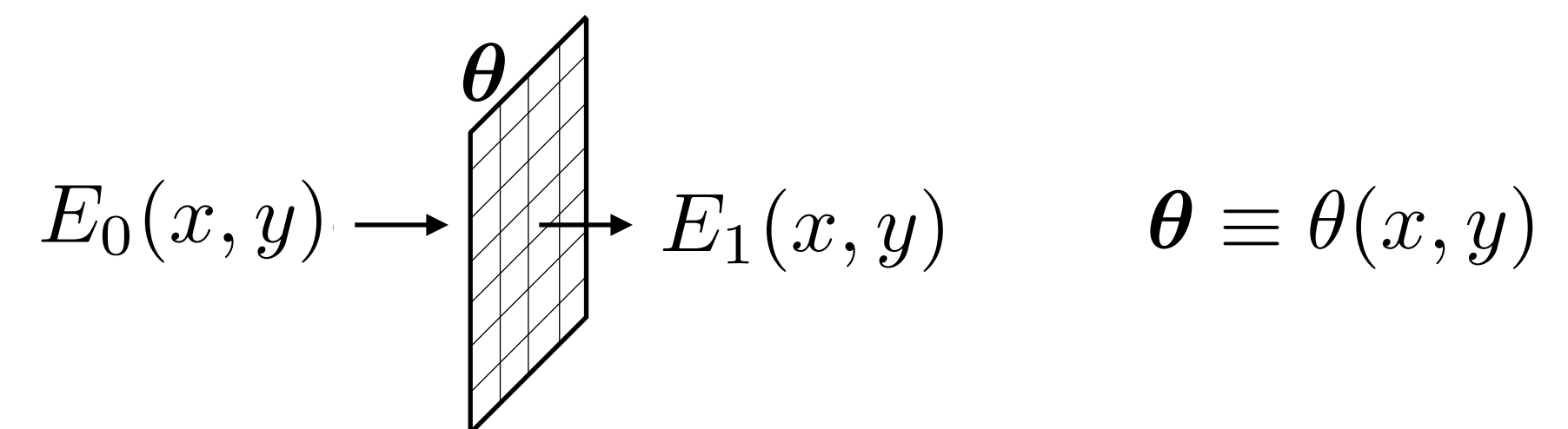
Coherent state pulse represented in (spatial) orthonormal basis $\psi_{n,m}(x, y)$

$$E_0(x, y) = \sum_{n,m} a_{n,m} \psi_{n,m}(x, y) \quad , \text{ with } \int \int \psi_{n,m}^*(x, y) \psi_{k,l}(x, y) dx dy \begin{cases} = 1, & n = k, \text{ and } m = l \\ = 0, & \text{otherwise} \end{cases}$$

Apply a spatially-varying phase $\theta(x, y)$: spatial-light modulator (SLM)

$$E_1(x, y) = E_0(x, y) e^{i\theta(x,y)} = b_{nm} \psi_{n,m}(x, y) \quad (\text{Represent it again in the chosen basis})$$

$$\sum_{n,m} a_{nm} \psi_{n,m}(x, y) e^{i\theta(x,y)} = \sum_{n,m} b_{nm} \psi_{n,m}(x, y)$$



$$\sum_{n,m} a_{nm} \underbrace{\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{n,m}(x, y) \psi_{k,l}^*(x, y) e^{i\theta(x,y)} dx dy \right]}_{U_{nm,kl}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n,m} b_{nm} \psi_{n,m}(x, y) \psi_{k,l}^*(x, y) dx dy = b_{kl}$$

- Multiplication by spatial phase is a mode transformation: $\mathbf{b} = U_{\theta} \mathbf{a}$

Recall: $E_1(x, y) = E_0(x, y)e^{i\theta(x, y)}$

Propagate the pulse $E_1(x, y)$ through a “4f system”: a Fourier Transform

$$E_1(x, y) \longrightarrow E_2(x', y') = \sum_{n, m} c_{nm} \psi_{n, m}(x', y') \quad \text{(Represent it again in the chosen basis)}$$

$$E_2(x', y') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_1(x, y) e^{-i2\pi(xx' + yy')} dx dy$$

Fourier propagation is a (fixed) mode transformation, $c = V b$

Repeating it many times...

$$E^{(\text{in})}(x, y) = \sum_{n, m} a_{nm}^{(\text{in})} \psi_{n, m}(x, y) \longrightarrow \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array} \dots \theta_N \longrightarrow E^{(\text{out})}(x', y') = \sum_{n, m} a_{nm}^{(\text{out})} \psi_{n, m}(x', y')$$

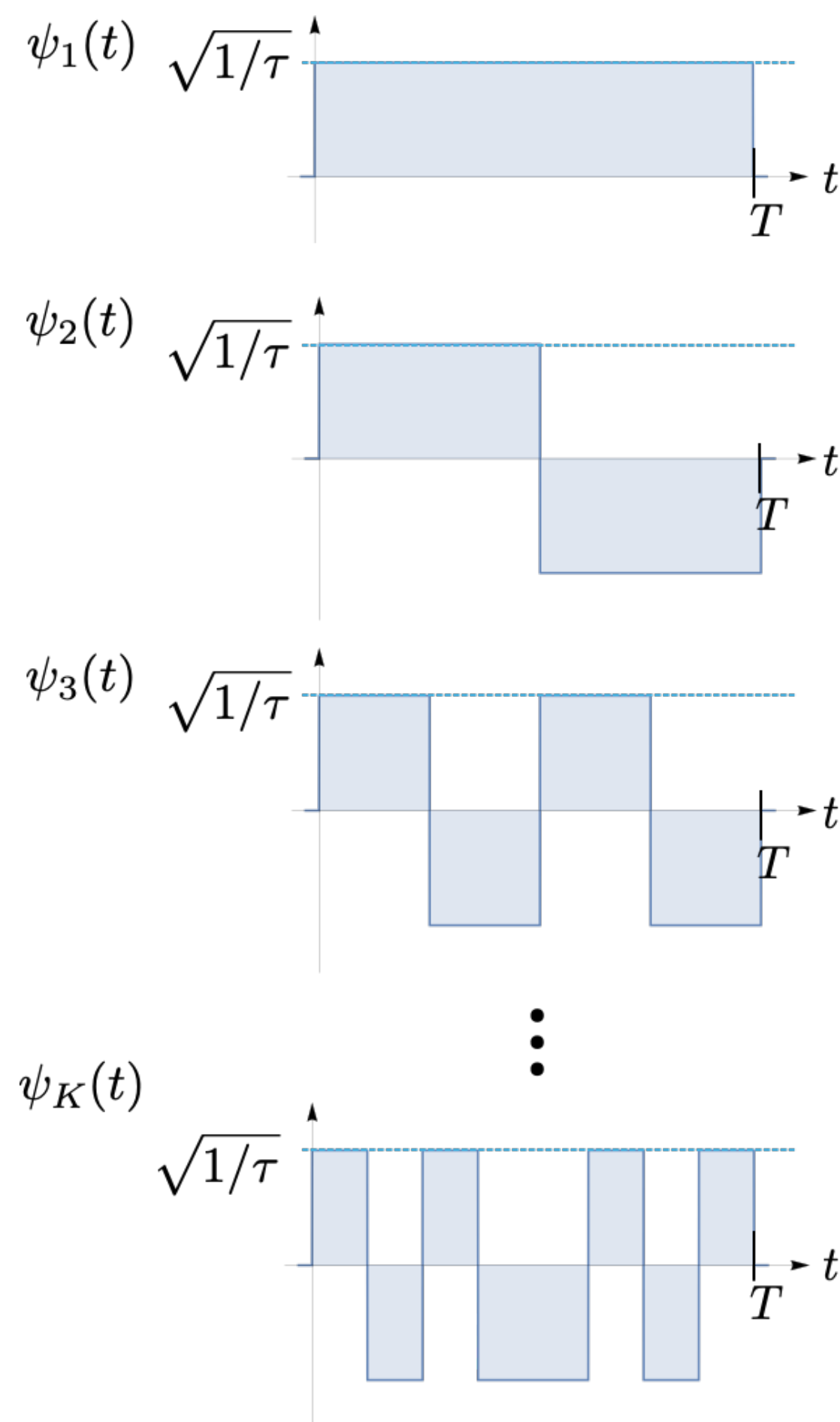
FT FT

$$\mathbf{a}^{(\text{out})} = U_{\theta_N} V \dots V U_{\theta_3} V U_{\theta_2} V U_{\theta_1} \mathbf{a}^{(\text{in})} = U \mathbf{a}^{(\text{in})}$$

Any unitary U can be approximated arbitrarily accurately by using a large enough number (N) of SLMs

Hadamard Coded Spatial Modes

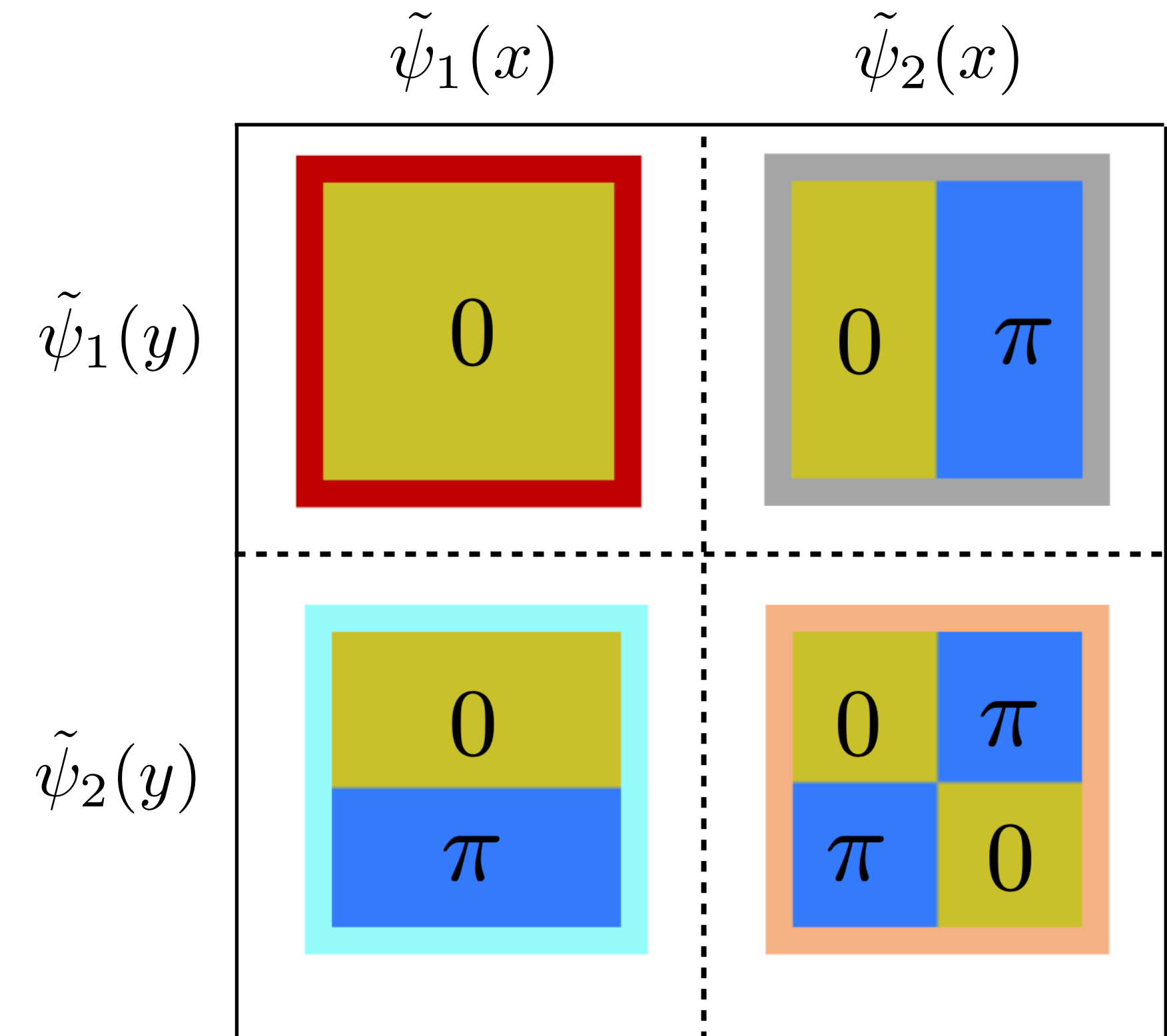
Recall: Hadamard code words
we had encountered when
talking about temporal modes



The spatial Hadamard modes are similarly defined

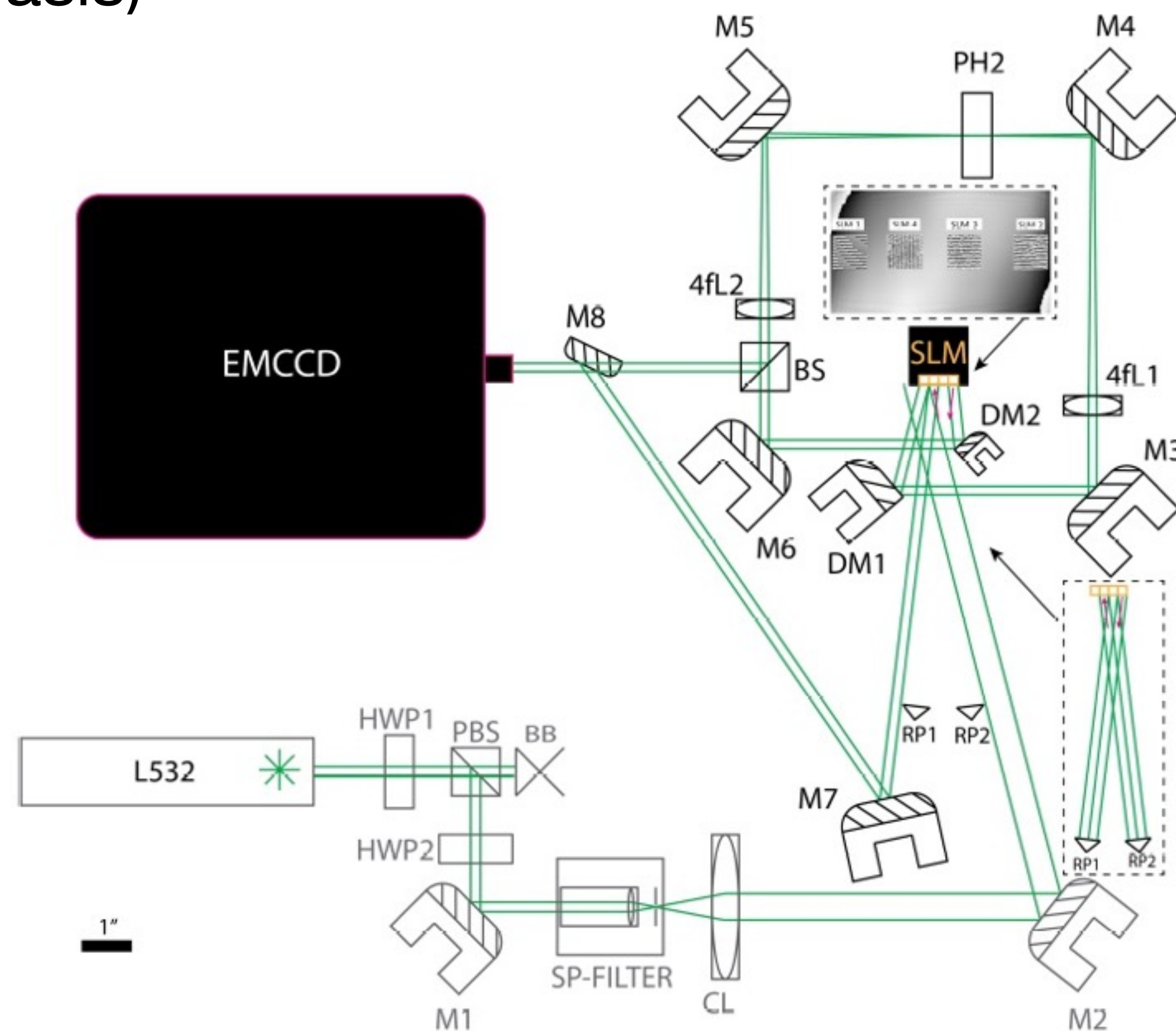
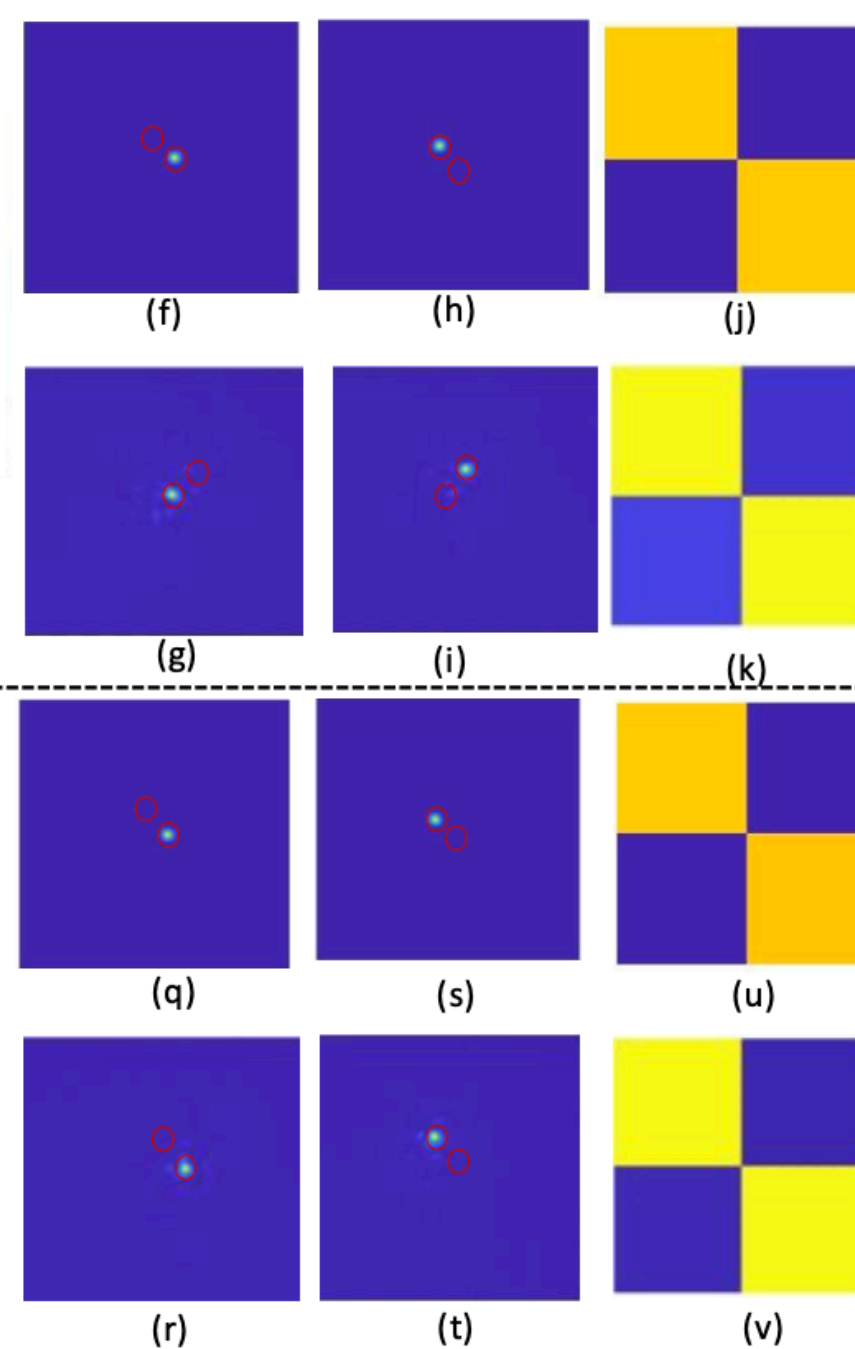
$$\psi_{n,m}(x, y) = \underbrace{\tilde{\psi}_n(x)}_{\text{Hadamard mode function in the } x \text{ axis variable}} \times \underbrace{\tilde{\psi}_m(y)}_{\text{Hadamard mode function in the } y \text{ axis variable}}$$

As an example, let
us consider the first
four spatial
Hadamard modes



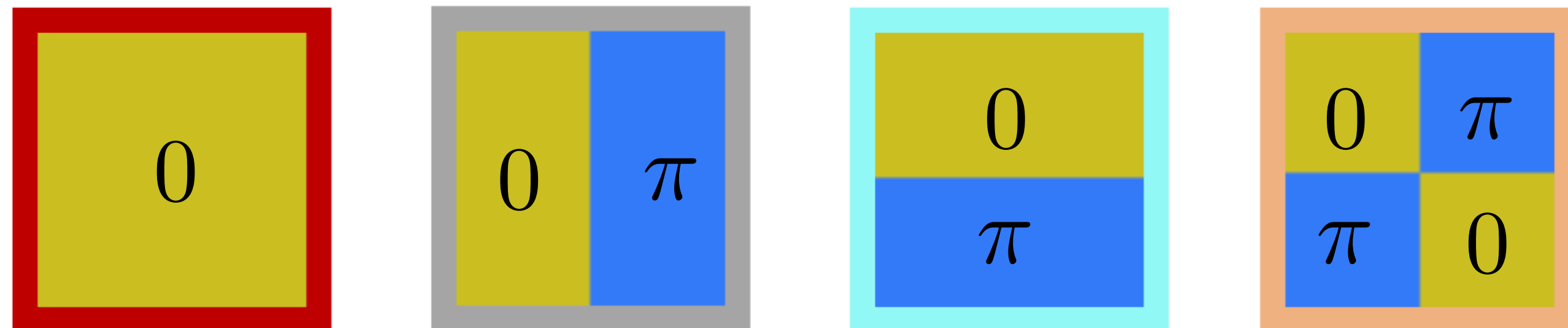
- Recall: Separating orthogonal modes $\psi_{n,m}(x, y)$ is a unitary mode transformation (from the chosen mode basis to the pixel basis)

Figure 1 displays a 3x3 grid of plots showing the evolution of a quantum state. The top row (a, b, c) shows the initial state and its evolution. The middle row (d, e, f) shows the state after a Hadamard gate. The bottom row (g, h, i) shows the state after a CNOT gate. The plots are color-coded from 0 to 2π , with a color bar at the bottom.

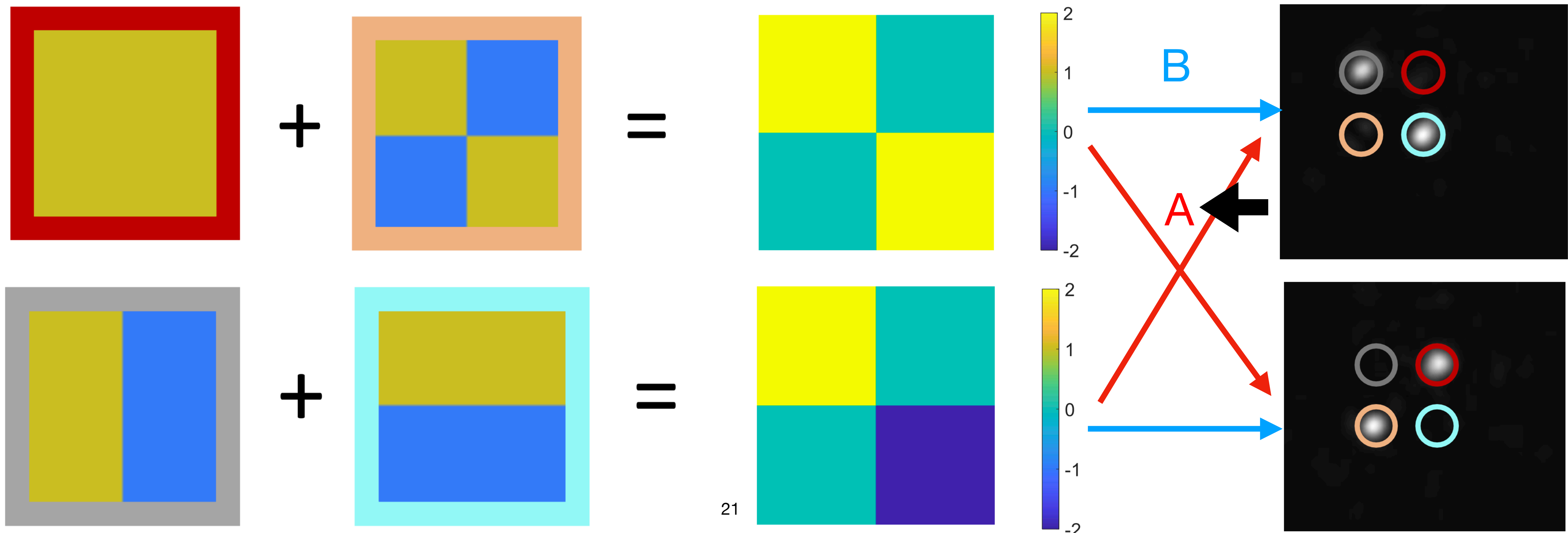


Exercise 3: Sorting spatial modes using MPLC

- MPLC to sort 4 of the 16 Hadamard-coded binary-phase spatial modes, shown:



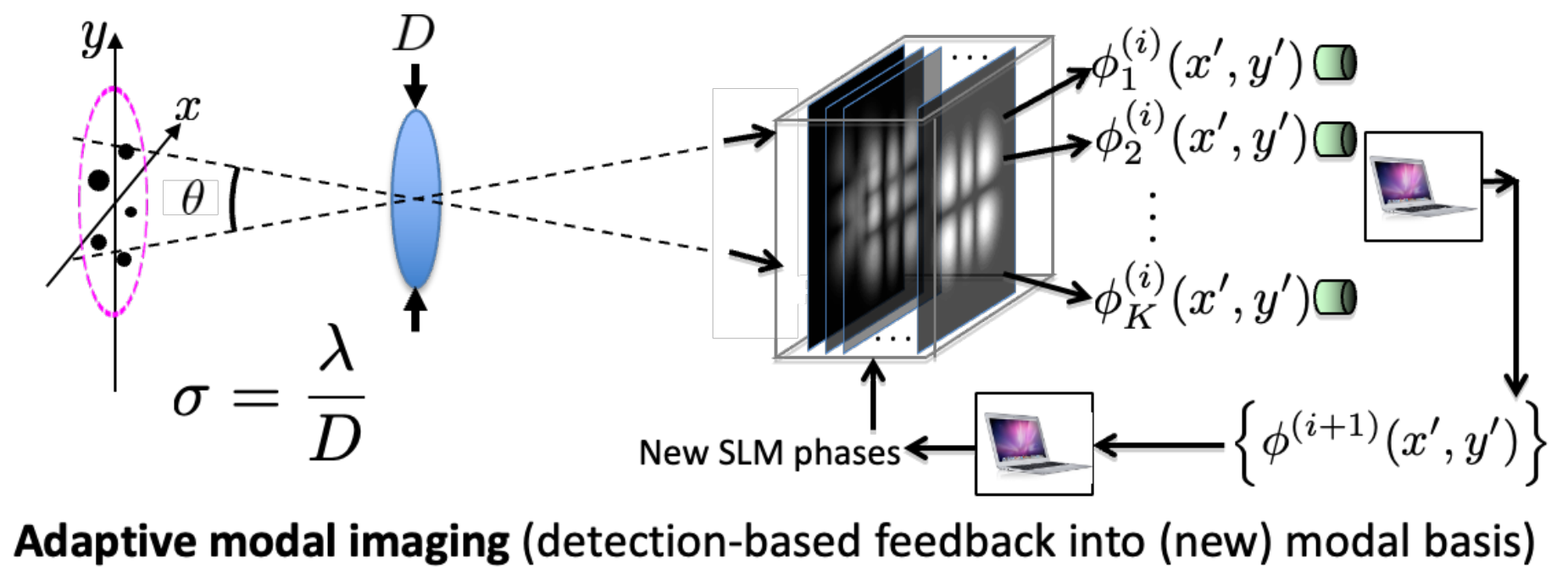
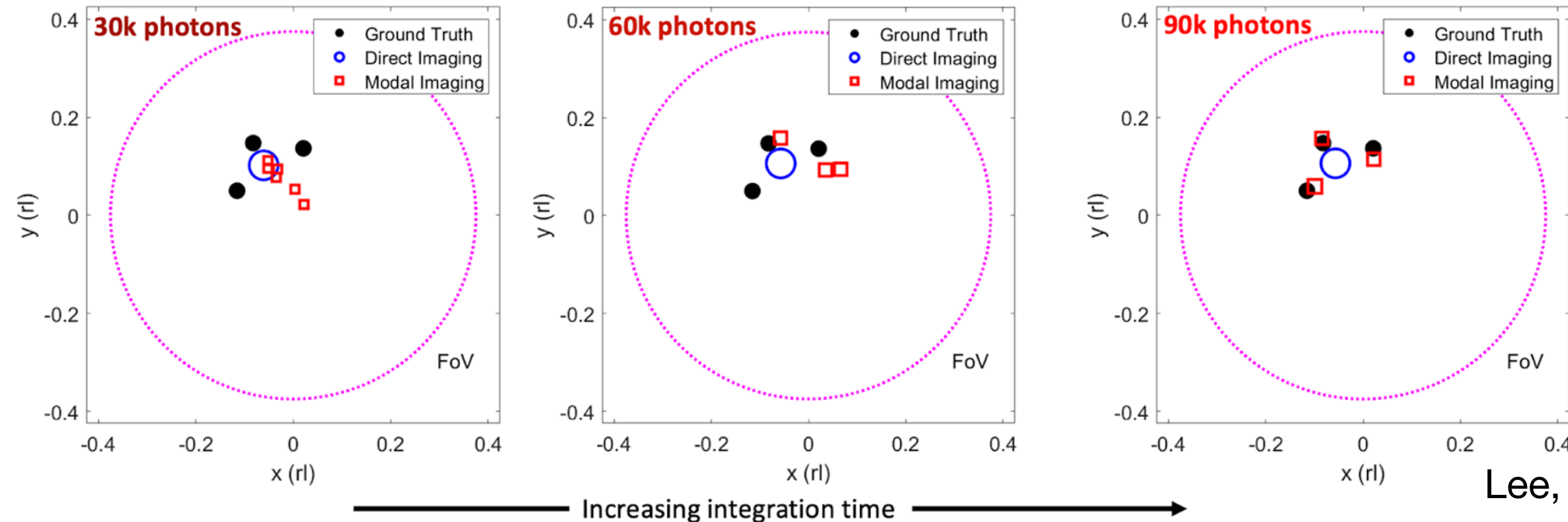
Which one (A or B) is the correct matching between the input field and the output intensity pattern at the output of the MPLC?



Applications of spatial mode transformations

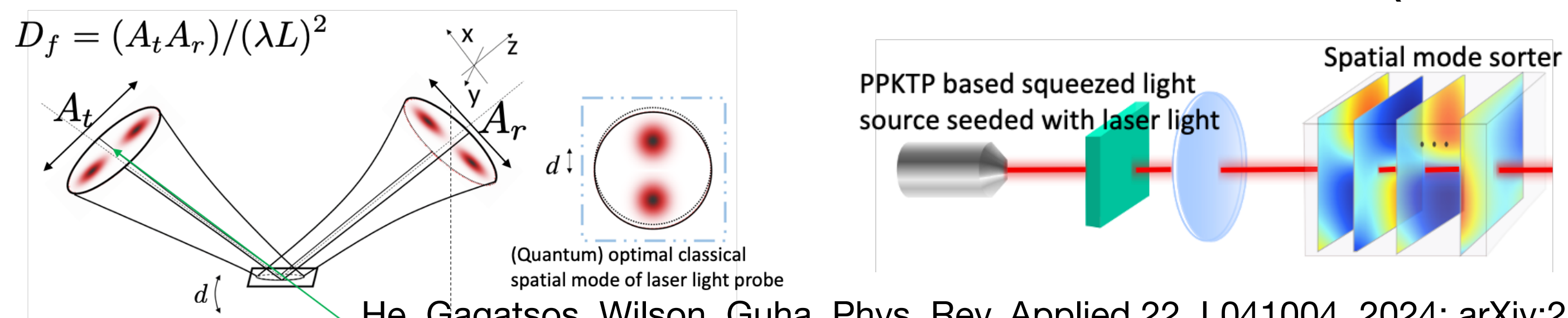
- Super-resolution imaging (astronomy, biology)

X and Y Angular lengths (of Field of View) in units of "Rayleigh Length", λ/D



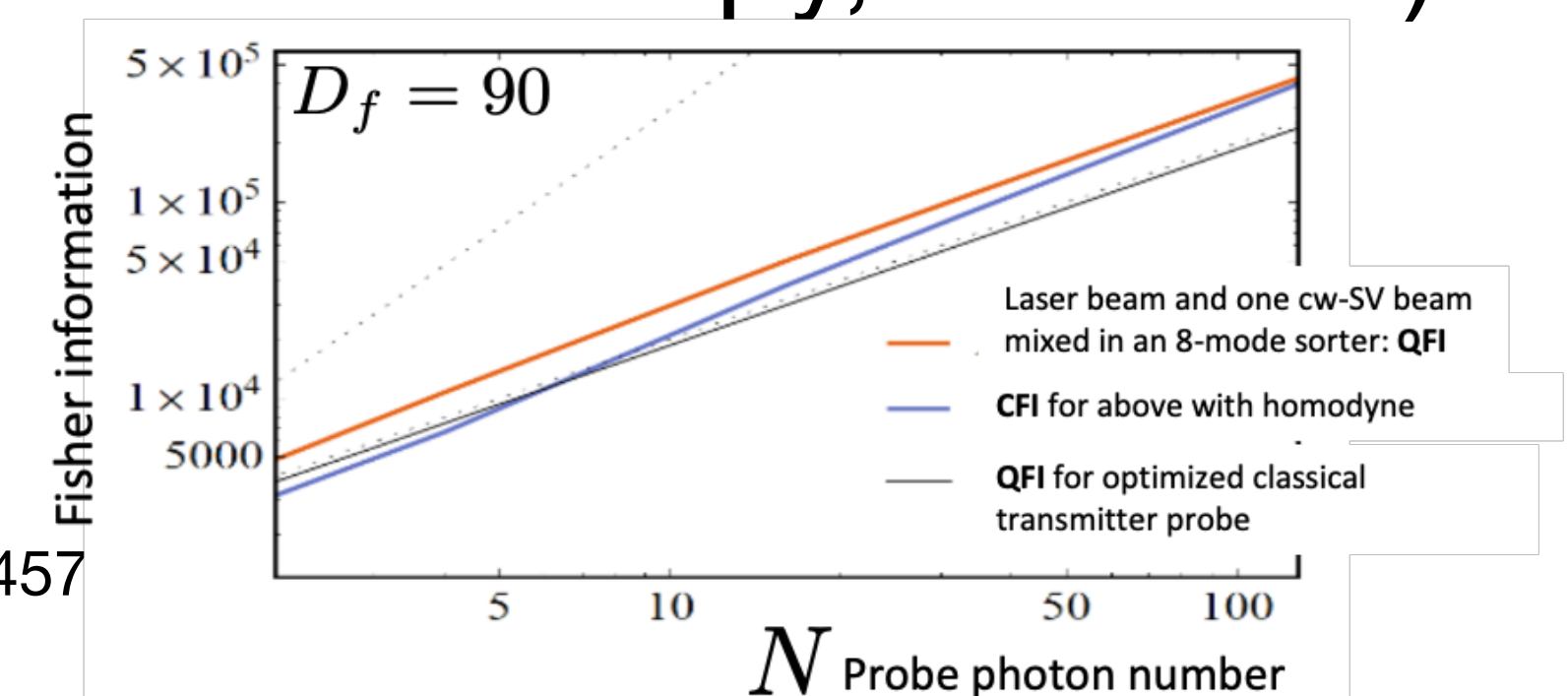
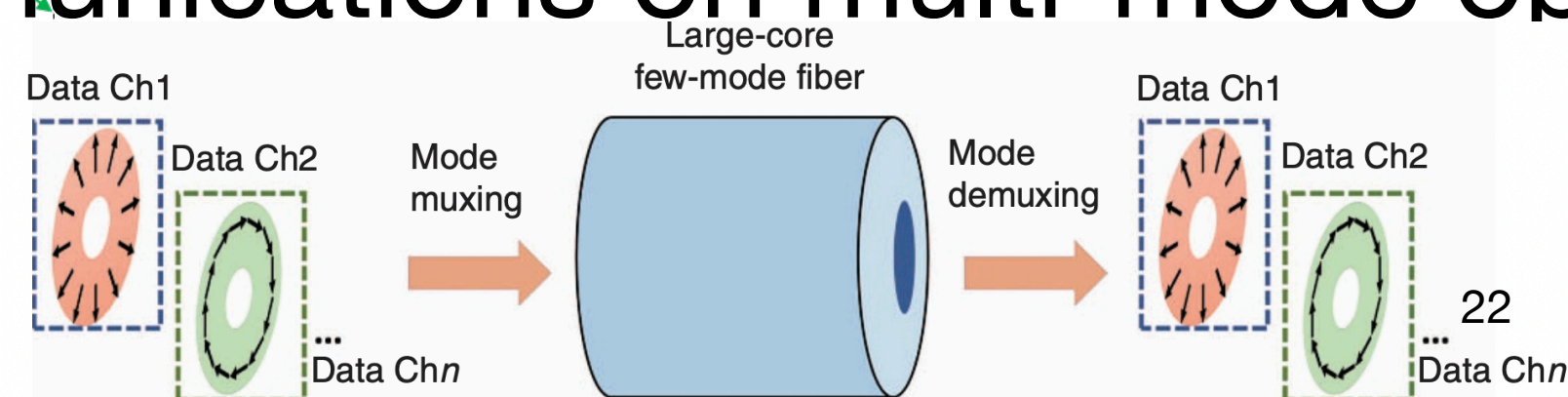
Lee, Gagatsos, Guha, Ashok, COSI 2022, IEEE JSTSP Vol 17, Issue 2 (2023)

- Generation of multi-spatial-mode entanglement from squeezed light, for high resolution beam-deflection measurements (atomic force microscopy, lasercom)



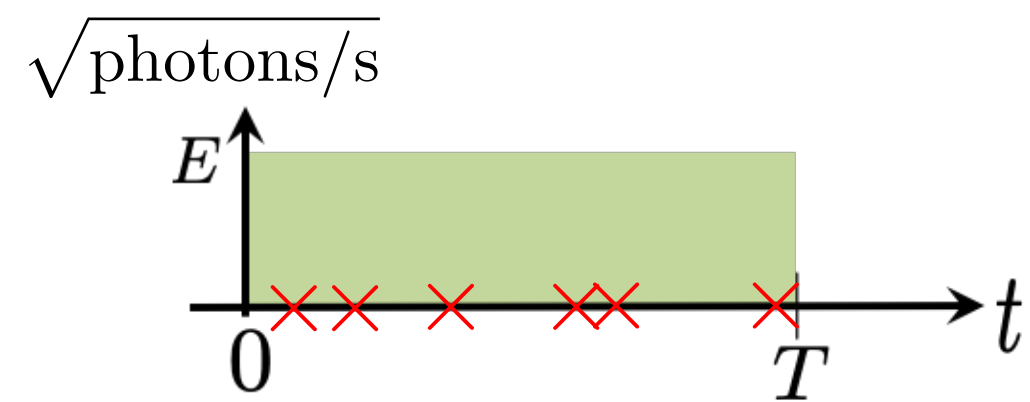
He, Gagatsos, Wilson, Guha, Phys. Rev. Applied 22, L041004, 2024; arXiv:2509.11457

- Laser communications on multi-mode optical fiber



Alan E Willner, Light: Science & Applications volume 7, page 18002 (2018)

Ideal photon detection on a coherent state



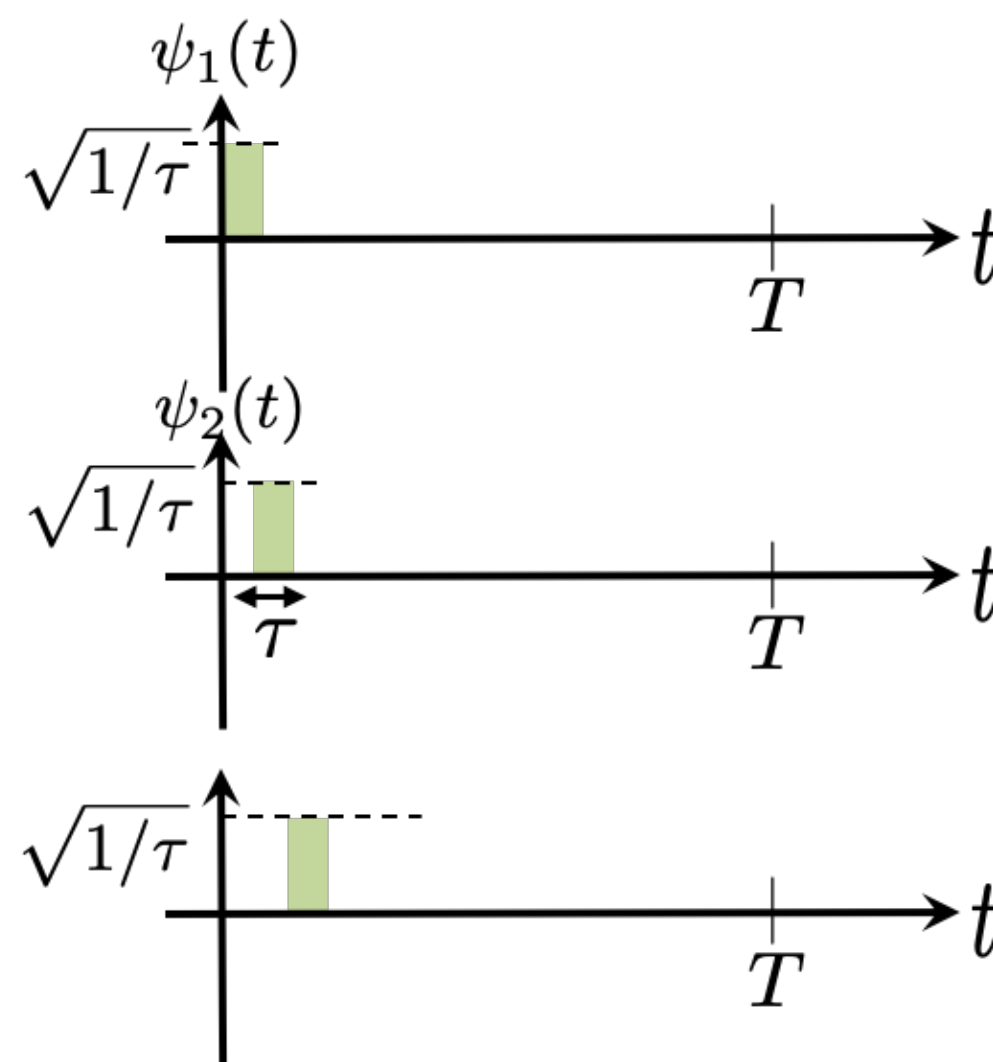
(1) Single-mode coherent state of the mode: $\phi(t)$
 $|\alpha\rangle$, $\alpha = \sqrt{N}$, $N = E^2 T$



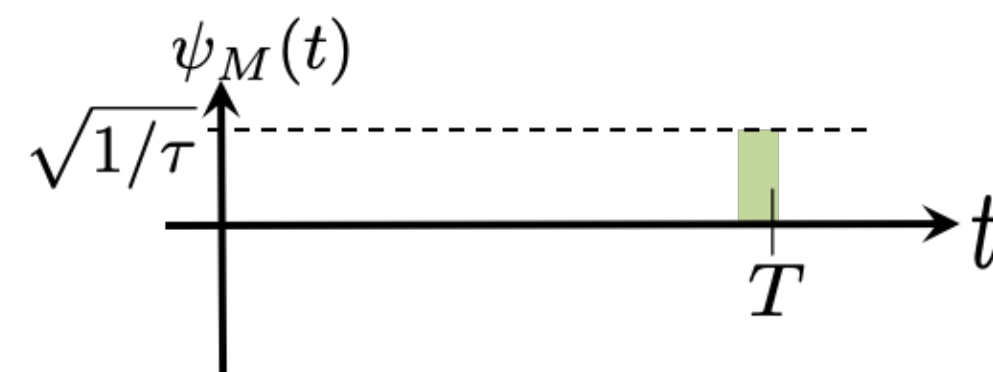
(2) M-mode coherent state of the modes: $\psi_k(t)$, $k = 1, \dots, M$

$$|\gamma\rangle|\gamma\rangle \dots |\gamma\rangle, \quad \gamma = \sqrt{\frac{N}{M}} \ll 1$$

$$M \gg 1$$



$M = \frac{T}{\tau}$ Orthogonal temporal modes



$$|\alpha\rangle \longrightarrow \text{Detector} \longrightarrow k$$

Recall: Probability of detecting k photons,

$$N = |\alpha|^2$$

$$p_k = \frac{e^{-N} N^k}{k!}$$

$$p_0 = e^{-N}, p_1 = N e^{-N}, p_2 = N^2 e^{-N} / 2, \dots$$

Probability of detecting k photons on coherent state $|\gamma\rangle$ of mode $\psi_i(t)$, $i = 1, 2, \dots, M$

$$p_k = \frac{e^{-N/M} (N/M)^k}{M \gg 1 \quad k!} = e^{-N/M}, k = 0 \text{ (no click)}$$

$$\approx 1 = e^{-N/M}, k = 1 \text{ (one click)}$$

Arrival rate (photons per second):

$$\lambda \equiv \frac{\gamma^2}{\tau} = \frac{N}{M\tau} = \frac{N}{T} = E^2$$

Poisson Point Process: Memoryless arrival process

Beamsplitters and interference

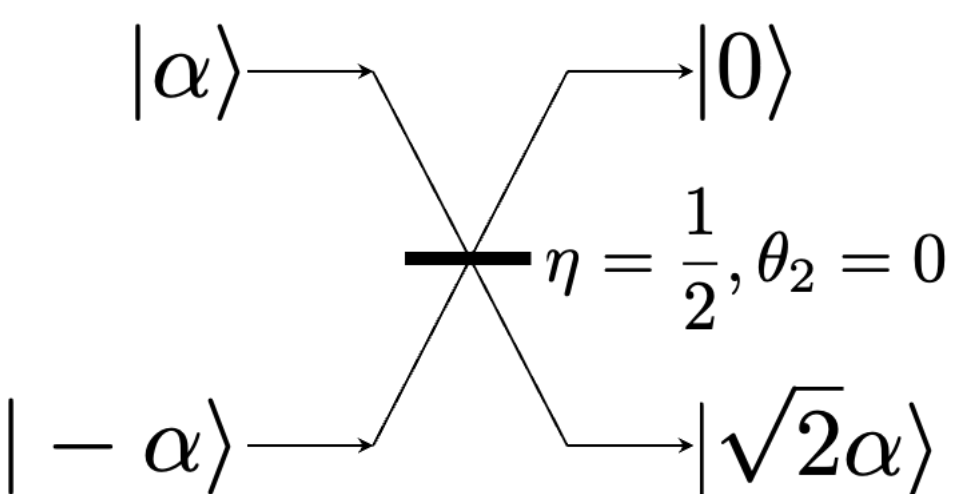
- Interfering two coherent states in a beamsplitter

$$\tau = \sqrt{\eta}, r = \sqrt{1 - \eta}$$

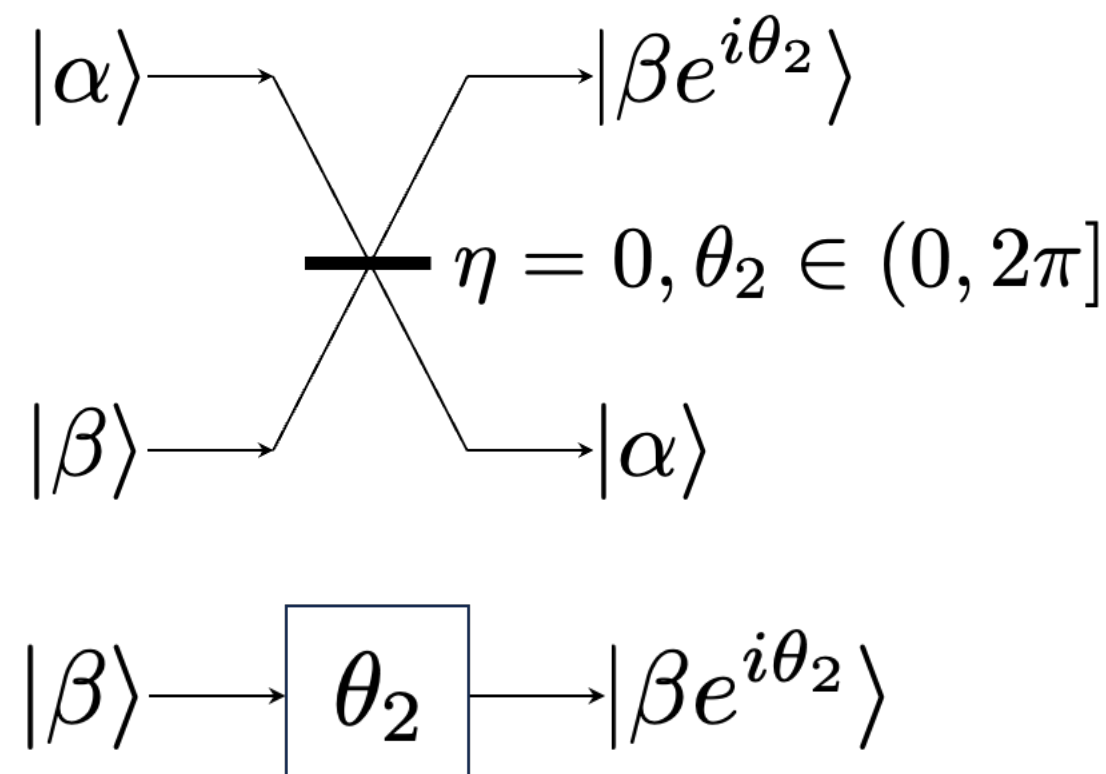
$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\eta} & e^{i\theta_2} \sqrt{1 - \eta} \\ \sqrt{1 - \eta} & -e^{i\theta_2} \sqrt{\eta} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

Transmissivity, $\eta \equiv \cos^2(\theta_1), \theta_1 \in (0, \pi]$
Phase, $\theta_2 \in (0, 2\pi]$

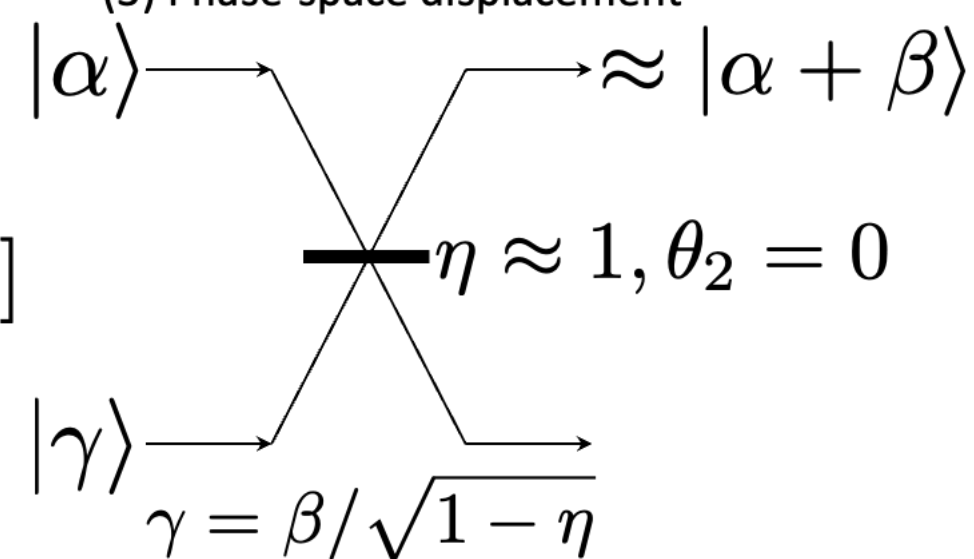
(1) Constructive vs. Destructive interference



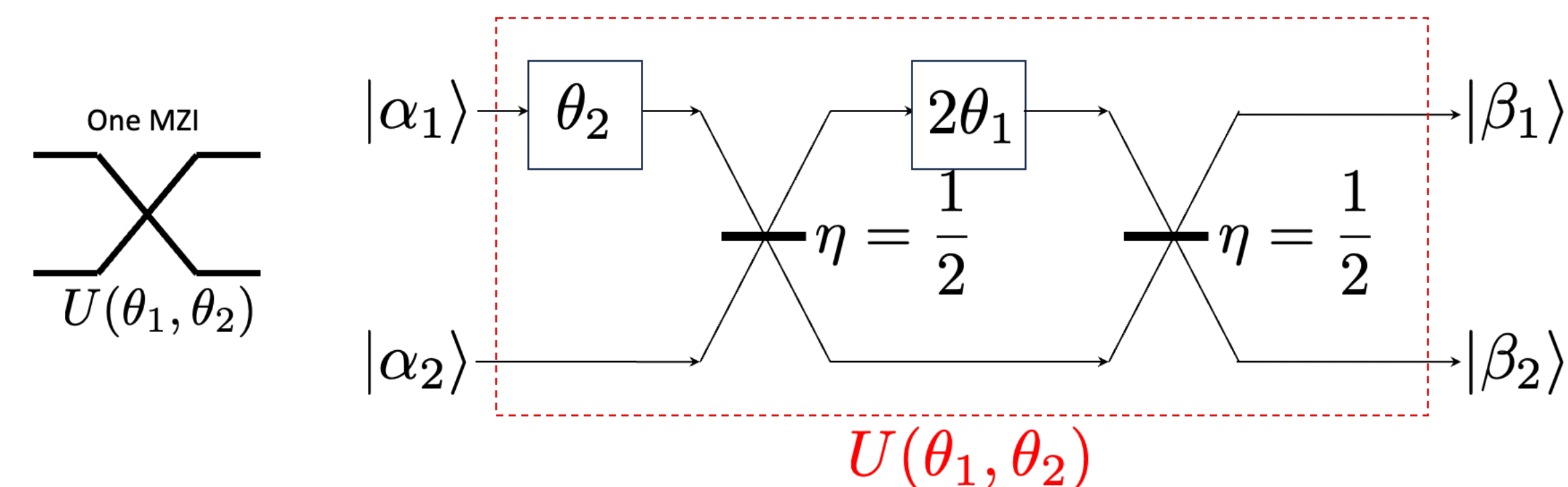
(2) Phase (pure reflection or pure transmission)



(3) Phase-space displacement



- The Mach-Zehnder Interferometer (MZI)

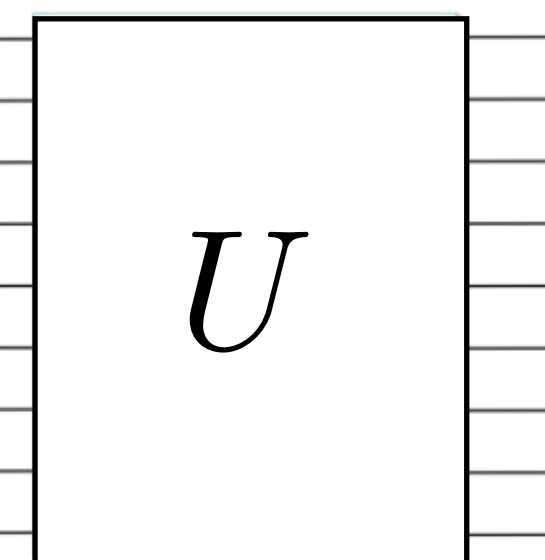


Reck et al., PRL 73, 1 (1994)

Clements et al., Optica 3 (12), 1460-1465 (2016)

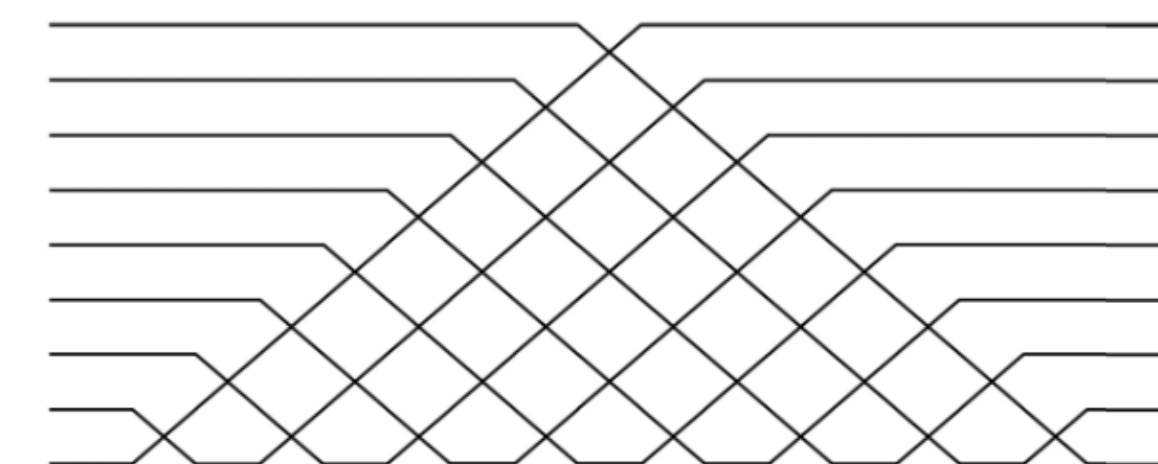
$\frac{n(n-1)}{2}$ MZIs

1
2
:
n

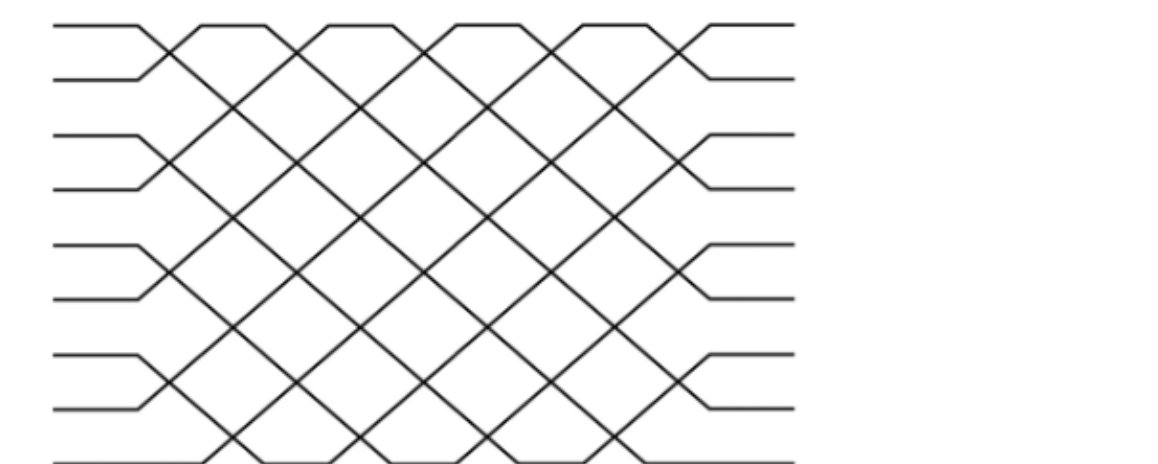


=

a

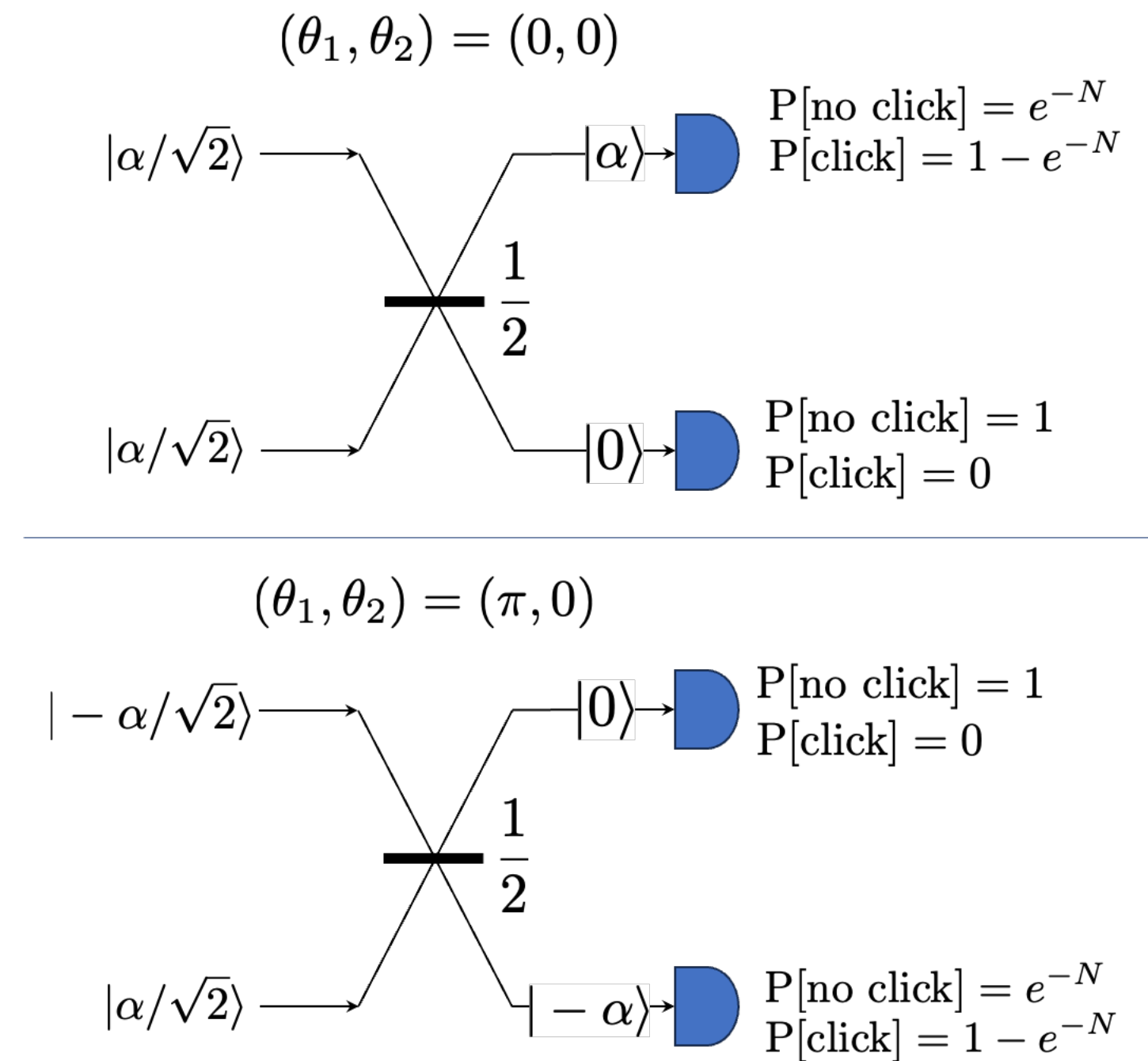
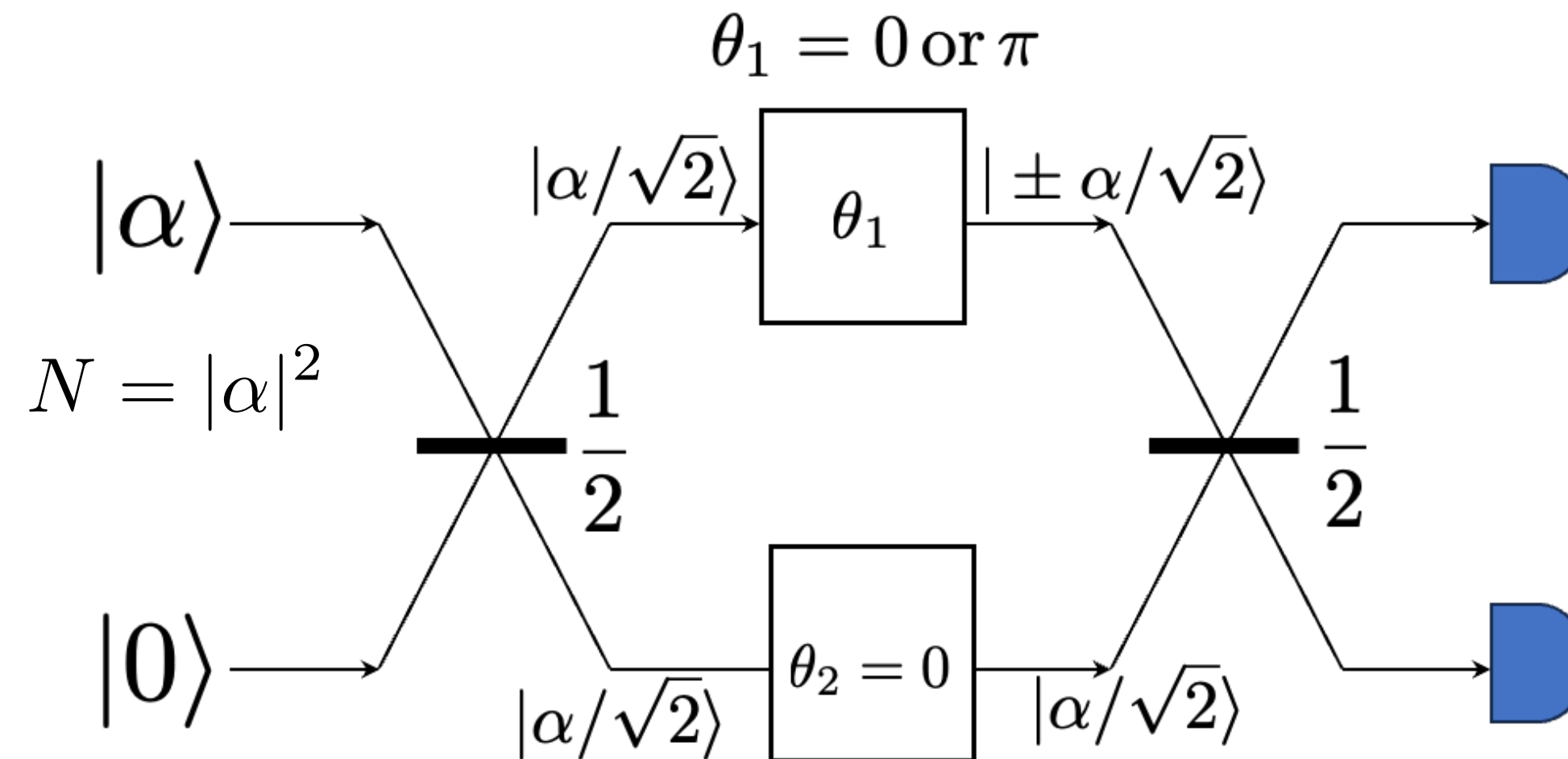
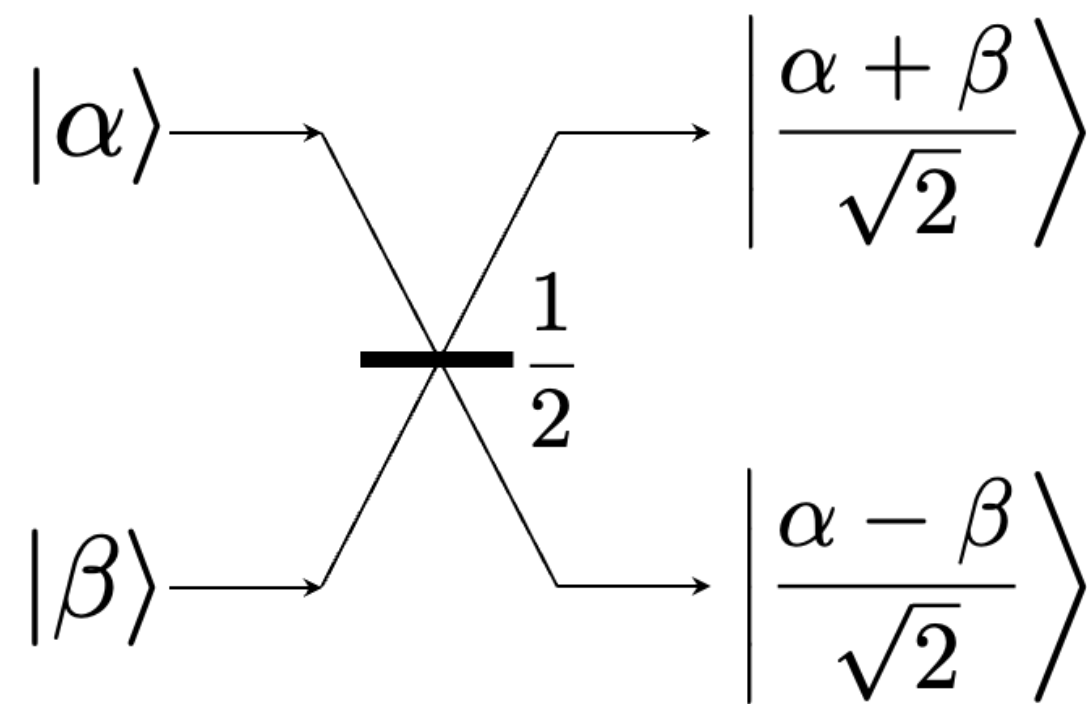


b



Exercise 4: Reading phase-coded information

Consider a 50-50 Beamsplitter:



• What is the probability of error for choosing between the two phase patterns (0, 0) versus (π, 0)?

• A. Zero

Let us assume that the prior probability (abbreviated “prior”) of each of the phase patterns 0,0 and π,0 (let us call them “codewords”) is 1/2, i.e., the occurrence of each pattern is equally likely.

• B. e^{-N}

• C. $1 - e^{-N}$

Then, the average probability of error is calculated using:

• D. $\frac{1}{2}e^{-N}$

[Probability of error given the phase pattern 0,0 actually occurred] X [Prior probability that 0,0 occurred, which is 1/2]
+
[Probability of error given the phase pattern π,0 actually occurred] X [Prior probability that π,0 occurred, which is 1/2]

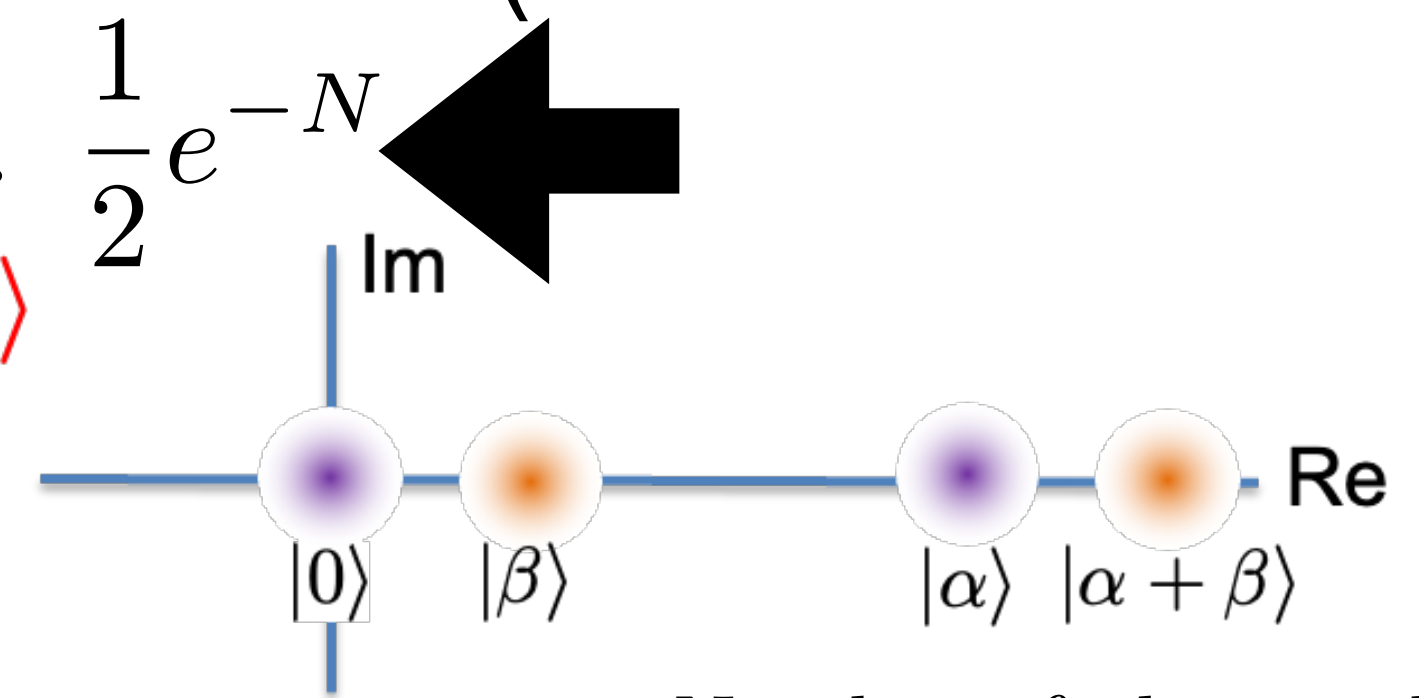
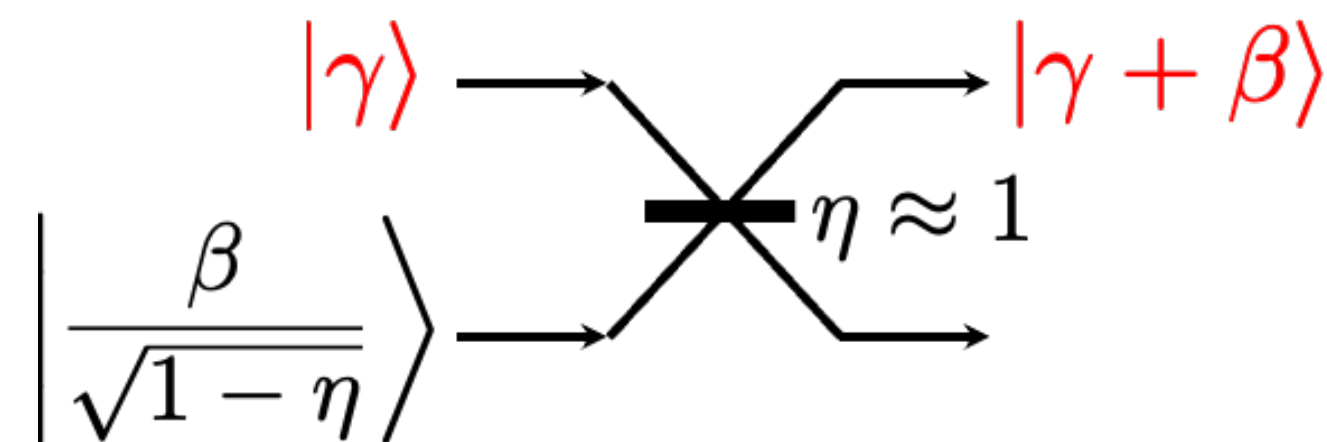
$$\left[(1 - e^{-N}) \times 0 \right] + \left[e^{-N} \times \frac{1}{2} \right]$$

Exercise 5: Receiver for optical communication

- Symbols of the on off keying (OOK) modulation “alphabet”: $\{|0\rangle, |\alpha\rangle\}$, $N = |\alpha|^2$
- Assuming equal priors, what is the (average) probability of error of choosing between the two OOK symbols using ideal photon detection (direct detection)?

A. Zero, B. e^{-N} , C. $1 - e^{-N}$ D. $\frac{1}{2}e^{-N}$ ←

- “Displace before detect”:



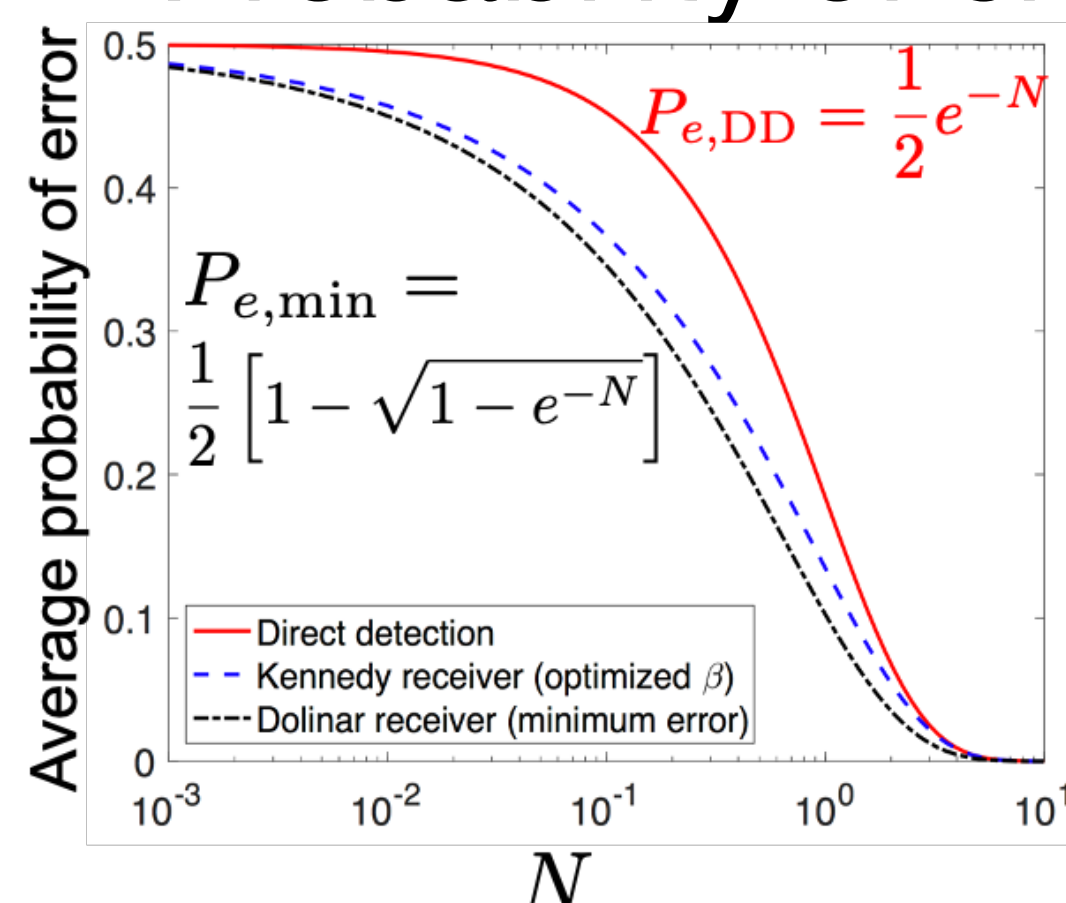
Number of photon clicks, k

- Probability of error, $P_e = P(H_1)P(H_2|H_1) + P(H_2)P(H_1|H_2)$

$$= \frac{1}{2}P(k > 0|H_1) + \frac{1}{2} \times P(k = 0|H_2) = \frac{1}{2} \left(1 - e^{-\beta^2}\right) + \frac{1}{2}e^{-(\alpha+\beta)^2}$$

Optimize (minimize) this over the choice of β

Plot the probability of error as a function of N



Hypothesis $H_1 : |0\rangle$

Hypothesis $H_2 : |\alpha\rangle$

Break

- Resume at 5:10 pm EST

Part C

Quantum States of Light

Introduction to Field Quantization



Maxwell's Equations

Source-free Maxwell's equations in vacuum

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

In vacuum (no magnetic or dielectric material), we have the constitutive relations

$$\mathbf{B} = \mu_0 \mathbf{H}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E}$$

μ_0 : Permeability of free space

ε_0 : Permittivity of free space

Permeability and permittivity define the speed of EM waves

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

E-field and B-field admit wave solutions

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$

Consider the frequency representation of an E-field

$$\mathbf{E}(\mathbf{r}, t) = \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega t} \boldsymbol{\mathcal{E}}^{(+)}(\mathbf{r}, \omega) + \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega t} \boldsymbol{\mathcal{E}}^{(-)}(\mathbf{r}, \omega)$$

We may write a mode expansion for each frequency component

$$\mathbf{r} = (x, y, z)$$

$$\boldsymbol{\mathcal{E}}^{(+)}(\mathbf{r}, \omega) = \underbrace{\sum_m}_{\text{Mode sum}} \underbrace{\mathcal{E} a^{(m)}(\omega)}_{\text{Amplitude}} \underbrace{\mathbf{u}^{(m)}(\mathbf{r}, \omega)}_{\text{Mode function ("Modes")}}$$

Mode index (label): m

Scale factor: $\mathcal{E}$

Modes : ortho-normal set of functions dictated by the geometry

$$\int d^3r \mathbf{u}^{(m)*}(\mathbf{r}, \omega) \mathbf{u}^{(k)}(\mathbf{r}, \omega) = \delta_{m,k}$$

Plane Wave Modes of Free Space

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = \mathbf{e}_m e^{-i\omega_m t} e^{i\mathbf{k}^{(m)} \cdot \mathbf{r}}$$

This is single plane wave with

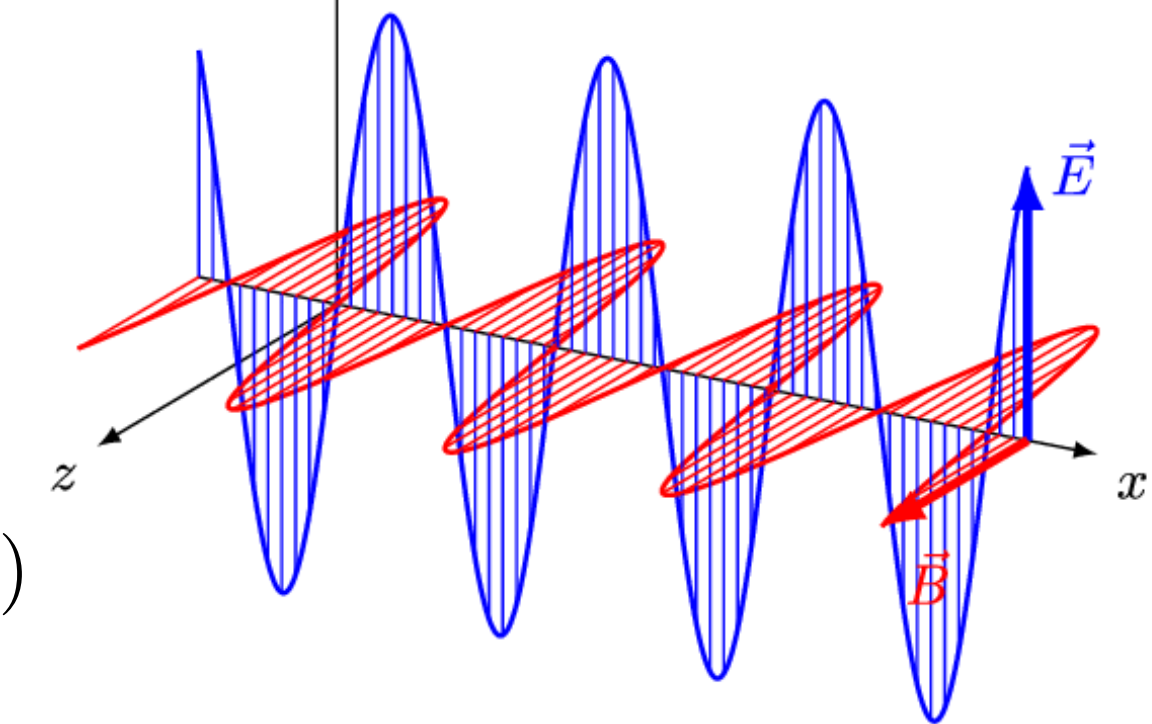
Frequency: ω_m

Field polarization direction: $\mathbf{e}_m$

Propagation direction: $\mathbf{k}^{(m)} = (k_x^{(m)}, k_y^{(m)}, k_z^{(m)})$

Where $\mathbf{k}^{(m)} \cdot \mathbf{r} = k_x^{(m)}x + k_y^{(m)}y + k_z^{(m)}z$

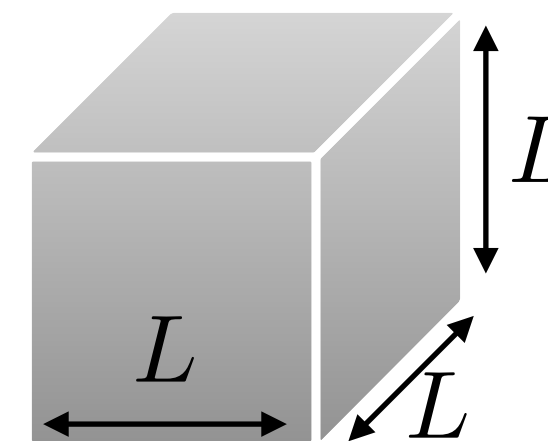
The allowed mode frequencies ω_m are continuous



E (B) field plane wave polarized along y(z) and propagating in the x direction

Modes of Highly Reflective Cavities

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = \sum_m a^{(m)} e^{-i\omega_m t} \mathbf{u}^{(m)}(\mathbf{r})$$



The allowed mode frequencies ω_m are discrete

If a the cavity is a cube of side length L

$$\mathbf{u}^{(m)}(\mathbf{r} + L \hat{\mathbf{x}}) = \mathbf{u}^{(m)}(\mathbf{r} + L \hat{\mathbf{y}}) = \mathbf{u}^{(m)}(\mathbf{r} + L \hat{\mathbf{z}}) = \mathbf{u}^{(m)}(\mathbf{r})$$

Why?

The highly reflective walls of the cavity will impose boundary conditions

$$\mathbf{u}^{(m)}(\mathbf{r}) = \mathbf{e}_m \exp \left[\frac{i2\pi}{L} (m_x x + m_y y + m_z z) \right]$$

Form a discrete set

Then we have the requirement for the wave vector

$$\mathbf{k}^{(m)} = \frac{2\pi}{L} (m_x, m_y, m_z)$$

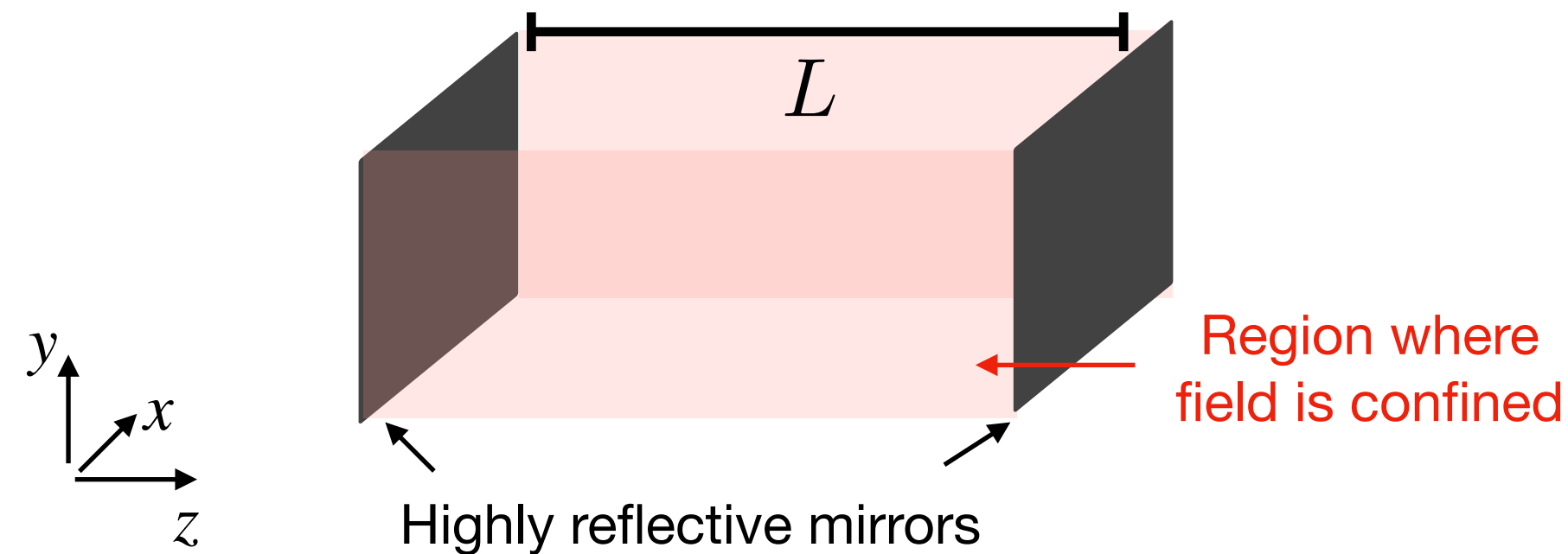
m_x, m_y, m_z : Integers that determine the overall mode index

This is a toy model of a cavity

Modes for Realistic Cavities

Light is 'not confined' along all three directions

Consider a plane-parallel cavity of length L



$$\mathbf{u}^{(m)}(\mathbf{r}, \omega) = \mathbf{e}^{(m)} \underbrace{f^{(m)}(x, y, \omega)}_{\text{Transverse mode}} \underbrace{\sin\left[\frac{\pi m z}{L}\right]}_{\text{Longitudinal mode}}$$

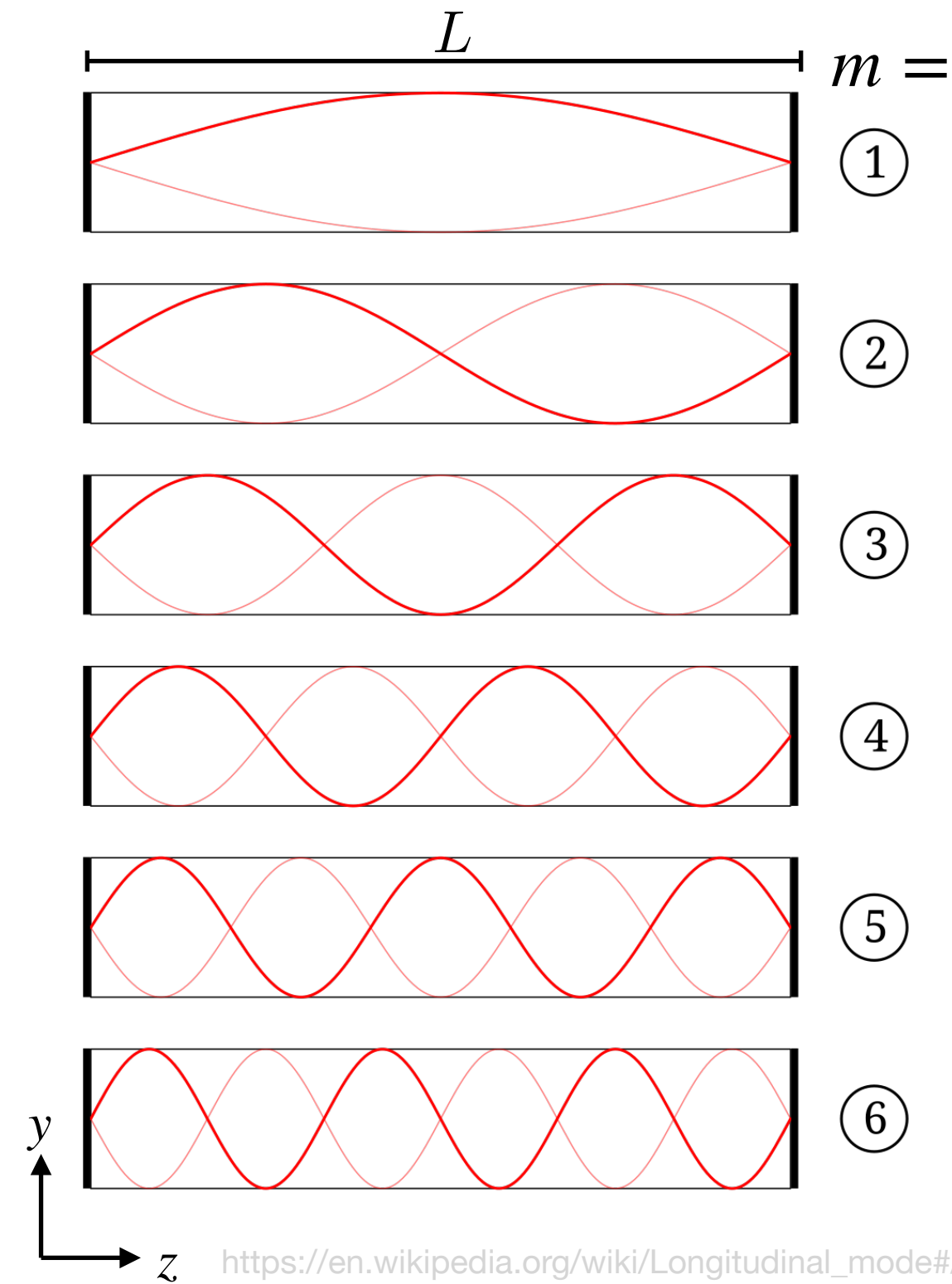
Transverse mode

Longitudinal mode

Longitudinal modes : different frequencies

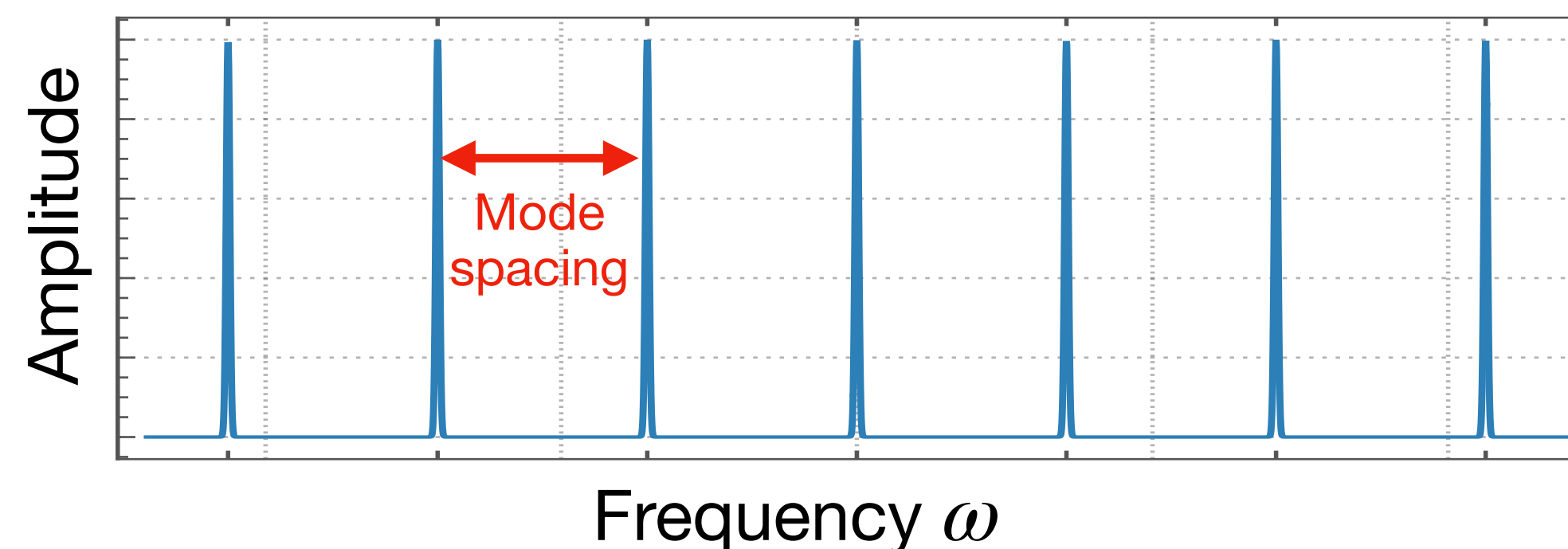
Transverse modes : orthogonal spatial functions ('patterns')

Longitudinal modes



https://en.wikipedia.org/wiki/Longitudinal_mode#/media/File:Longitudinal_mode_v2.svg

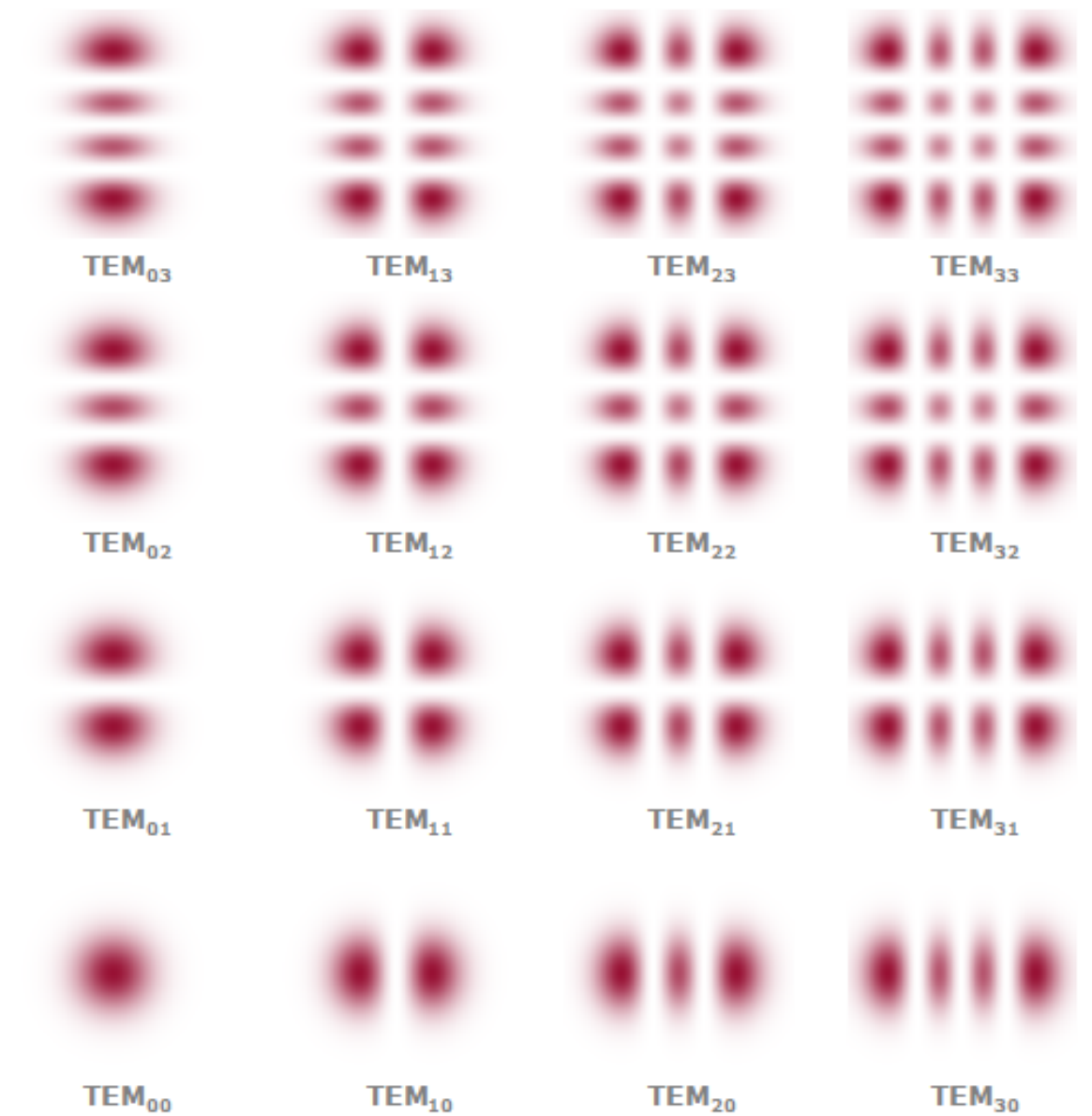
Longitudinal mode spectrum (structure)



Hermite-Gauss tranverse modes



$$f^{(m)}(x, y, \omega) \propto H_m(x)e^{-x^2} H_m(y)e^{-y^2}$$



https://www.rp-photonics.com/img/tem_modes.webp

Mode spacing :

$$\omega_m - \omega_{m-1} = \frac{\pi c}{L}$$

Quantization of the Electromagnetic Field

$$\mathbf{E}^{(+)}(\mathbf{r}, t) = \sum_m \int_0^\infty \mathcal{E}_m a^{(m)}(\omega) e^{-i\omega t} \mathbf{u}^{(m)}(\mathbf{r}, \omega) \longrightarrow \hat{\mathbf{E}}^{(+)}(\mathbf{r}, t) = \sum_m \int_0^\infty \mathcal{E}_m(\omega) \hat{a}_m(\omega) e^{-i\omega t} \mathbf{u}^{(m)}(\mathbf{r}, \omega)$$

Classical field description *Quantized field description*

Quantization rules
complex amplitude $\rightarrow$ operators

$a^{(m)}(\omega) \rightarrow \hat{a}_m(\omega)$ Defined as the 'lowering operator' (annihilation)

$a^{(m)*}(\omega) \rightarrow \hat{a}_m^\dagger(\omega)$ Defined as the 'raising operator' (creation)

Amplitude

$$\mathcal{E}_m \rightarrow \mathcal{E}_m(\omega) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 c}}$$

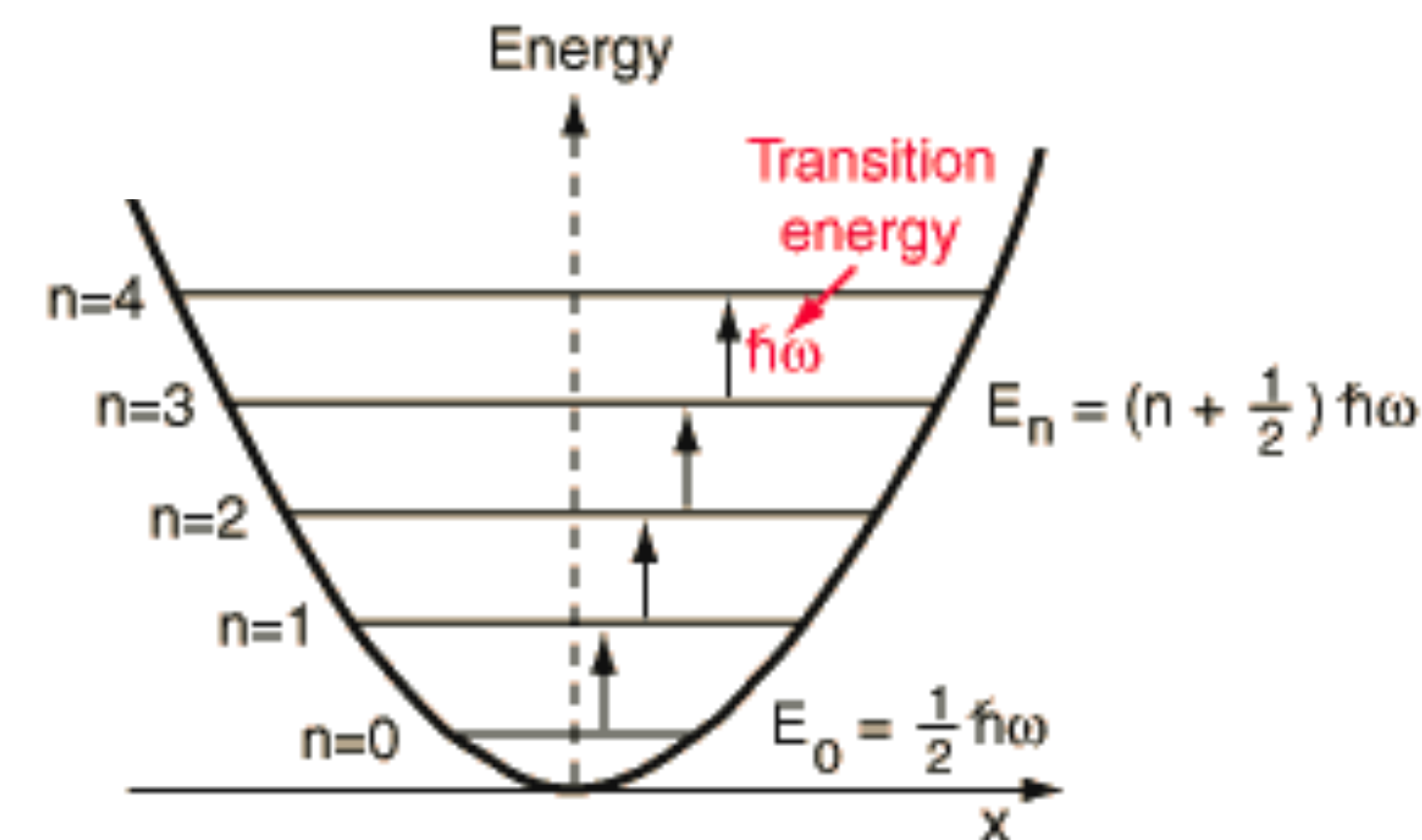
Canonical commutation rules

$$\begin{aligned} [\hat{a}_m(\omega), \hat{a}_k^\dagger(\omega)] &= \hat{a}_m(\omega) \hat{a}_k^\dagger(\omega) - \hat{a}_k^\dagger(\omega) \hat{a}_m(\omega) \\ &= \delta_{m,k} \end{aligned}$$

Hamiltonian for the mode

$$\hat{H}_m = \hbar\omega_m \left(\hat{a}_m^\dagger(\omega) \hat{a}_m(\omega) + \frac{1}{2} \right)$$

Resembles the quantum harmonic oscillator Hamiltonian



Eigenstates of the Hamiltonian operator

Noting the commutation relations of creation and annihilation operators with the Hamiltonian

$$[\hat{H}_m, \hat{a}_m] = \hbar\omega_m \left[\left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right), \hat{a}_m \right] \\ = -\hbar\omega_m \hat{a}_m$$

$$[\hat{H}_m, \hat{a}_m^\dagger] = \hbar\omega_m \hat{a}_m^\dagger$$

Hence, if we consider an eigenstate of the Hamiltonian operator

$$\hat{H}_m |\psi\rangle_m = E_n |\psi\rangle_m$$



$$\hat{H}_m \hat{a}_m |\psi\rangle_m = (E_n - \hbar\omega_m) \hat{a}_m |\psi\rangle_m$$

$$\hat{H}_m \hat{a}_m^\dagger |\psi\rangle_m = (E_n + \hbar\omega_m) \hat{a}_m^\dagger |\psi\rangle_m$$

$\hat{a}_m |\psi\rangle_m, \hat{a}_m^\dagger |\psi\rangle_m$ are eigenvectors as well - with lowered or raised eigenvalues

Energy difference: $\hbar\omega_m$

We assume the ground state $|\psi_0\rangle_m$ satisfies

$$\hat{a}_m |\psi_0\rangle_m = 0 \quad |\psi_0\rangle_m \neq 0$$

Eigenvalue:

$$\hat{H}_m |\psi_0\rangle_m = \frac{\hbar\omega_m}{2} |\psi_0\rangle_m$$

Fock (number) states

$$\{|n\rangle_m\}; \quad n \in \mathbb{Z}^+$$

Eigenvalue relation

$$\hat{H}_m |n\rangle_m = \hbar\omega_m \left(n + \frac{1}{2} \right) |n\rangle_m$$

Lowering and raising operators

$$\hat{a}_m |n\rangle_m = \sqrt{n} |n-1\rangle_m$$

$$\hat{a}_m^\dagger |n\rangle_m = \sqrt{n+1} |n+1\rangle_m$$

Eigenstates of the annihilation operator : Coherent States

$$\hat{a}_m |\alpha\rangle_m = \alpha |\alpha\rangle_m$$

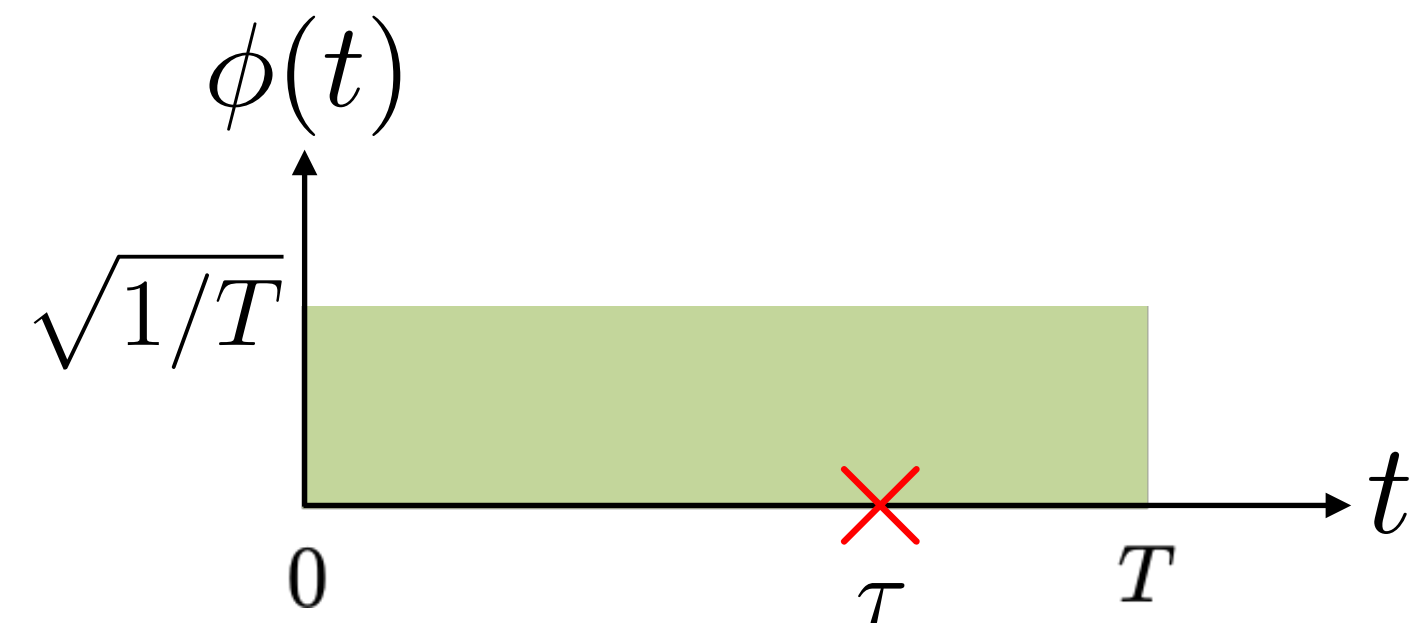
$$= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_m$$

Quantum description of classical laser light

Intuitively, these are quantum states of light where the annihilation operator and complex amplitude are 'interchangeable'

Number (Fock) state of a mode

- Single photon (Fock) state of a mode $\phi(t)$, detected by ideal photon detector



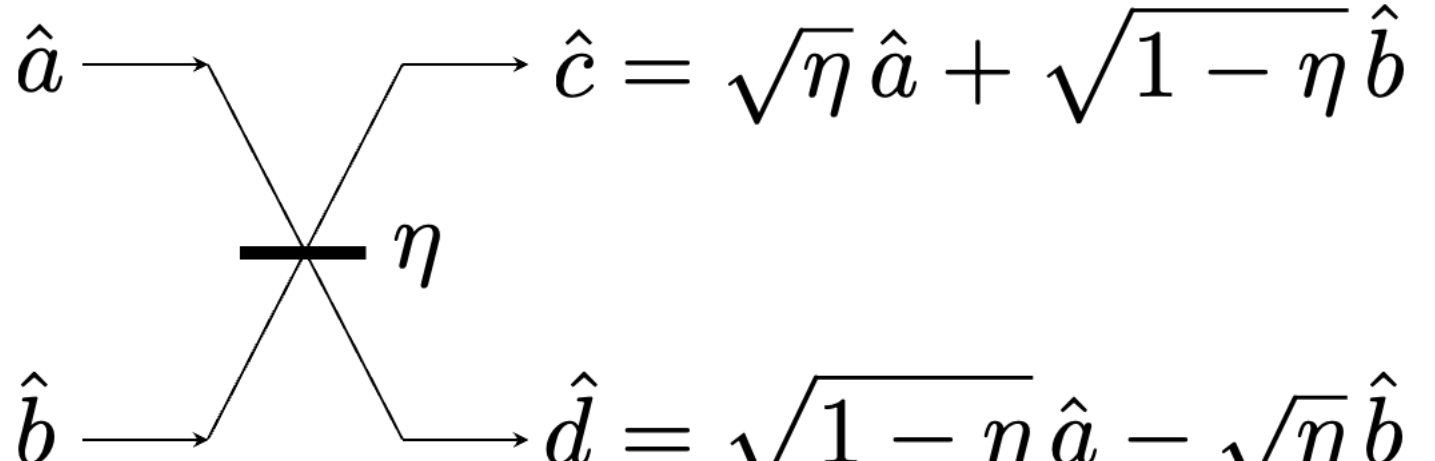
$$|1\rangle_{\phi(t)}$$

Precisely a single click will be detected, at (a random) time τ whose probability distribution $p(\tau) = |\phi(\tau)|^2$

- n -photon Fock state of the above mode: $|n\rangle_{\phi(t)}$
 - Ideal photon detection will produce precisely n clicks
 - Click arrival process is no longer “memoryless” (unlike the Poisson Point Process for coherent states). Why?
 - Arrival (or not) of one photon click in a time interval changes the probability distribution of the remaining clicks

Interfering number states in a beamsplitter

- A beamsplitter transforms the annihilation and creation operators of the two interfering modes [“Heisenberg Picture” of QM]:



$$\begin{aligned} \hat{c} &= \sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{b} & \hat{a}|n\rangle &= \sqrt{n} |n-1\rangle \\ \hat{d} &= \sqrt{1-\eta} \hat{a} - \sqrt{\eta} \hat{b} & \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \Rightarrow (a^\dagger)^n |0\rangle = \sqrt{n!} |n\rangle \\ & & [\hat{a}, \hat{a}^\dagger] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \hat{I} \end{aligned}$$

Inverse Transform

$$\begin{aligned} \hat{a} &= \sqrt{\eta} \hat{c} + \sqrt{1-\eta} \hat{d} \\ \hat{b} &= \sqrt{1-\eta} \hat{c} - \sqrt{\eta} \hat{d} \end{aligned}$$

- Excite the input modes in number states $|n\rangle$ and $|m\rangle$; what is the output state?

$$|\psi\rangle = |n\rangle |m\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} \frac{(\hat{b}^\dagger)^m}{\sqrt{m!}} |0\rangle |0\rangle = \frac{(\sqrt{\eta} \hat{c}^\dagger + \sqrt{1-\eta} \hat{d}^\dagger)^n}{\sqrt{n!}} \frac{(\sqrt{1-\eta} \hat{c}^\dagger - \sqrt{\eta} \hat{d}^\dagger)^m}{\sqrt{m!}} |0\rangle |0\rangle$$

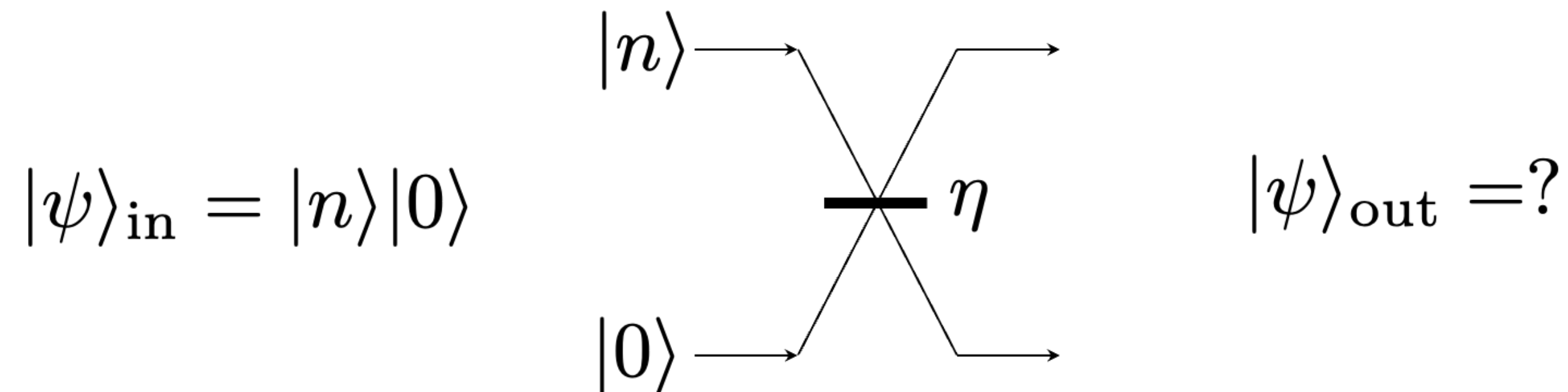
- Examples: Quantum state of the (a, b) modes, $|\psi\rangle_{\text{in}}$ Quantum state of the (c, d) modes, $|\psi\rangle_{\text{out}}$

$$\eta = \frac{1}{2} \quad |\psi\rangle = |1\rangle |0\rangle = a^\dagger |0\rangle |0\rangle = \left(\frac{c^\dagger + d^\dagger}{\sqrt{2}} \right) |0\rangle |0\rangle = \frac{|1\rangle |0\rangle + |0\rangle |1\rangle}{\sqrt{2}} \quad \text{“Entangled states”}$$

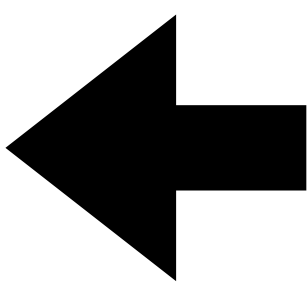
$$\eta = \frac{1}{2} \quad |\psi\rangle = |1\rangle |1\rangle = a^\dagger b^\dagger |0\rangle |0\rangle = \left(\frac{c^\dagger + d^\dagger}{\sqrt{2}} \right) \left(\frac{c^\dagger - d^\dagger}{\sqrt{2}} \right) |0\rangle |0\rangle = \frac{1}{2} (c^{\dagger 2} - d^{\dagger 2}) |0\rangle |0\rangle = \frac{|2\rangle |0\rangle - |0\rangle |2\rangle}{\sqrt{2}}$$

Exercise 6: Interfering number states in a beamsplitter

- Splitting an n photon number state in a beamsplitter (with vacuum in other port)



A. $|\psi\rangle_{\text{out}} = \frac{|n\rangle|0\rangle + |0\rangle|n\rangle}{\sqrt{2}}$

B. $|\psi\rangle_{\text{out}} = \sum_{k=0}^n \sqrt{\binom{n}{k} \eta^k (1-\eta)^{n-k}} |k\rangle |n-k\rangle$ 

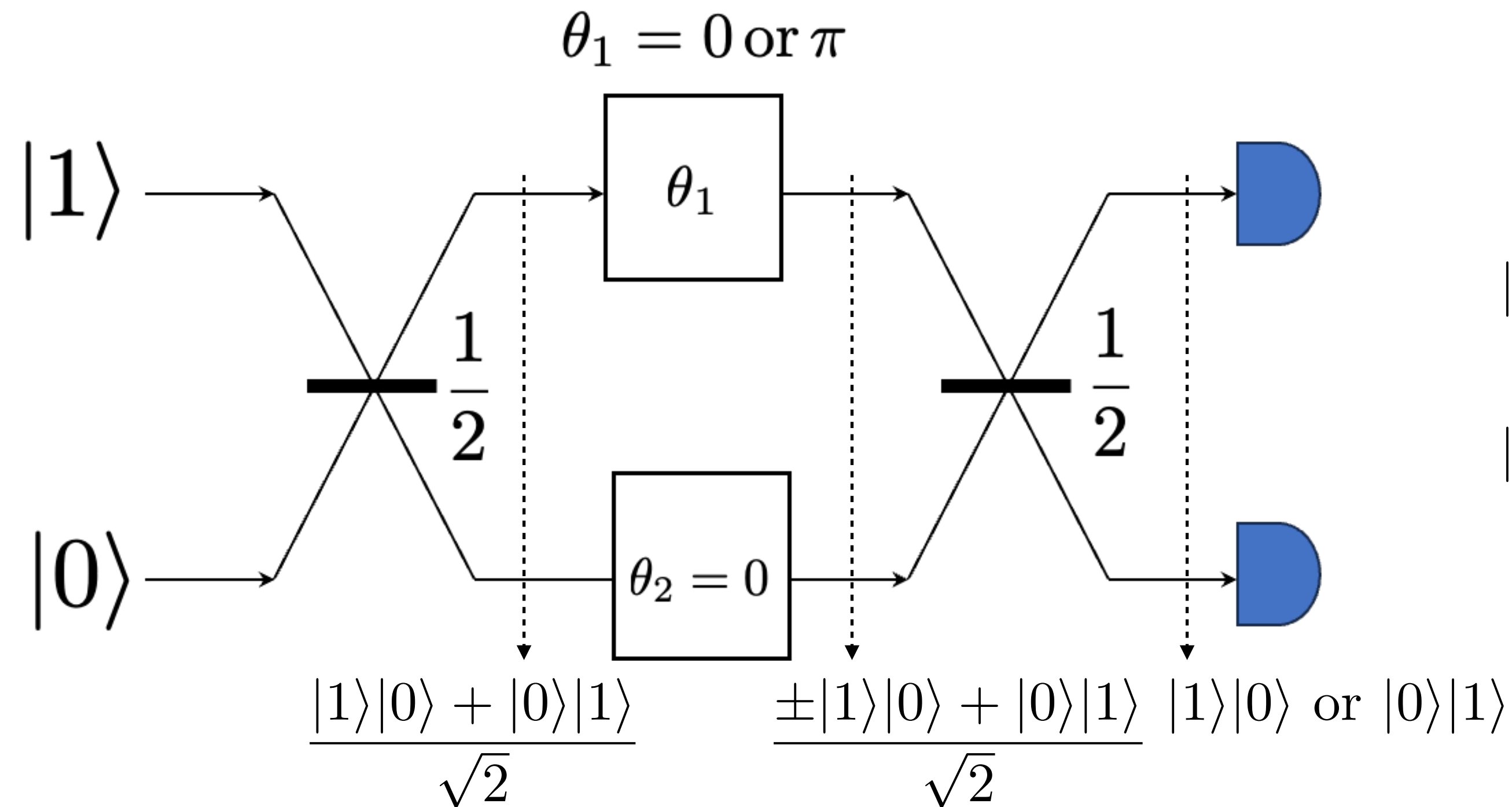
C. $|\psi\rangle_{\text{out}} = \sum_{k=0}^n \sqrt{\binom{n}{k} \eta^k (1-\eta)^{n-k}} |n-k\rangle |k\rangle$

D. $|\psi\rangle_{\text{out}} = \frac{|n\rangle|0\rangle - |0\rangle|n\rangle}{\sqrt{2}}$

- Note: multi-mode unitary transformation of a single mode number state generates entanglement across multiple orthogonal modes

Exercise 7: Reading phase-coded information using number state

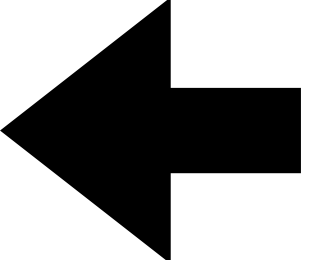
$N = 1$ photon



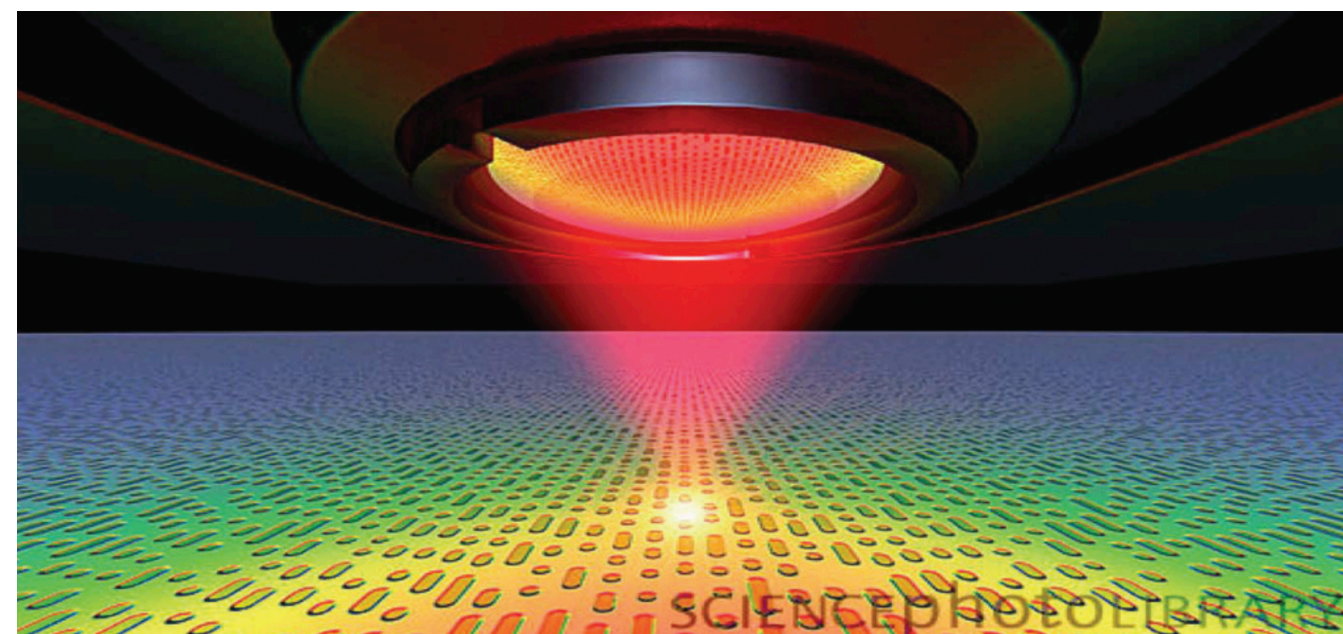
$$|1\rangle |0\rangle \rightarrow \frac{|1\rangle |0\rangle + |0\rangle |1\rangle}{\sqrt{2}}$$

$$|0\rangle |1\rangle \rightarrow \frac{|1\rangle |0\rangle - |0\rangle |1\rangle}{\sqrt{2}}$$

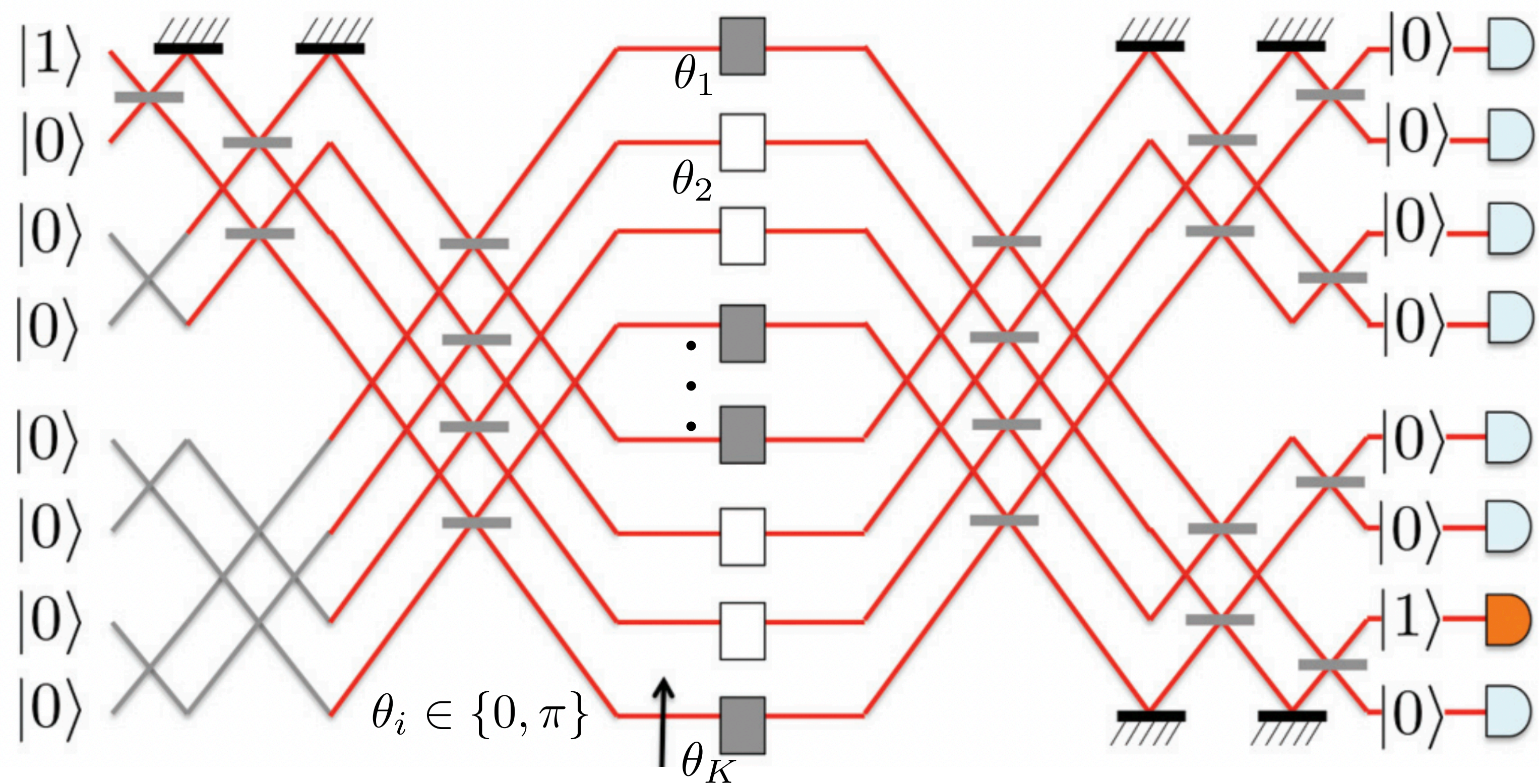
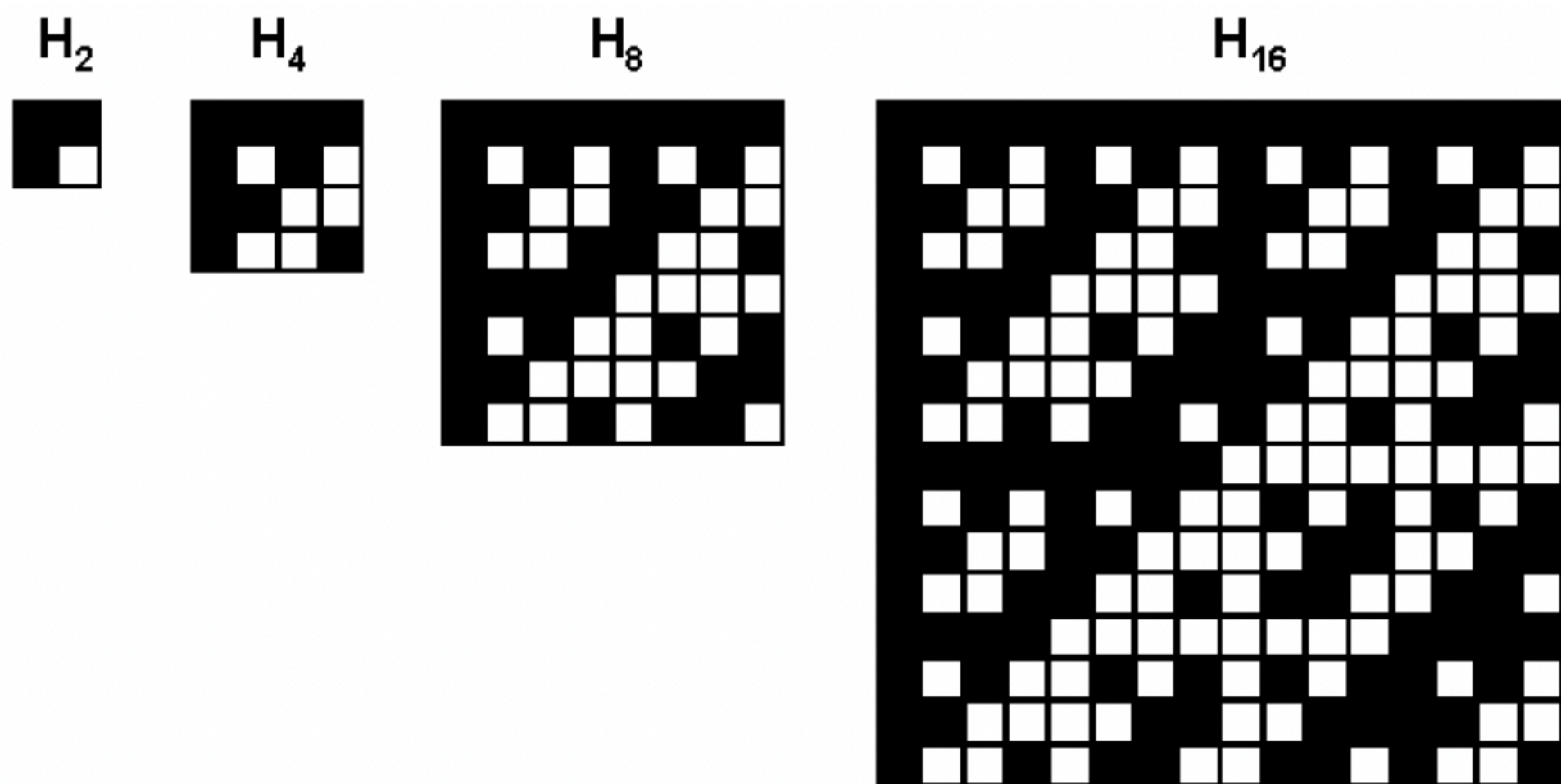
• What is the probability of error for choosing between the two phase patterns (0, 0) versus (π , 0)?

- A. Zero 
- B. e^{-N}
- C. $1 - e^{-N}$
- D. $\frac{1}{2}e^{-N}$

Reading boundless bits using a single photon



Optical
reading of
information
from a CD
or a DVD
disk



$$|\psi\rangle = (|100\dots 0\rangle + |010\dots 0\rangle + \dots + |000\dots 1\rangle) / \sqrt{K}$$

Depending upon which of the $K=8$ codewords the light hit, the photon will show up at one of the 8 detectors (with probability 1), thereby enabling reading $\log_2(K)$ bits of information with zero error probability. This is not possible using any classical light probe (including coherent states).

Part D

State Discrimination

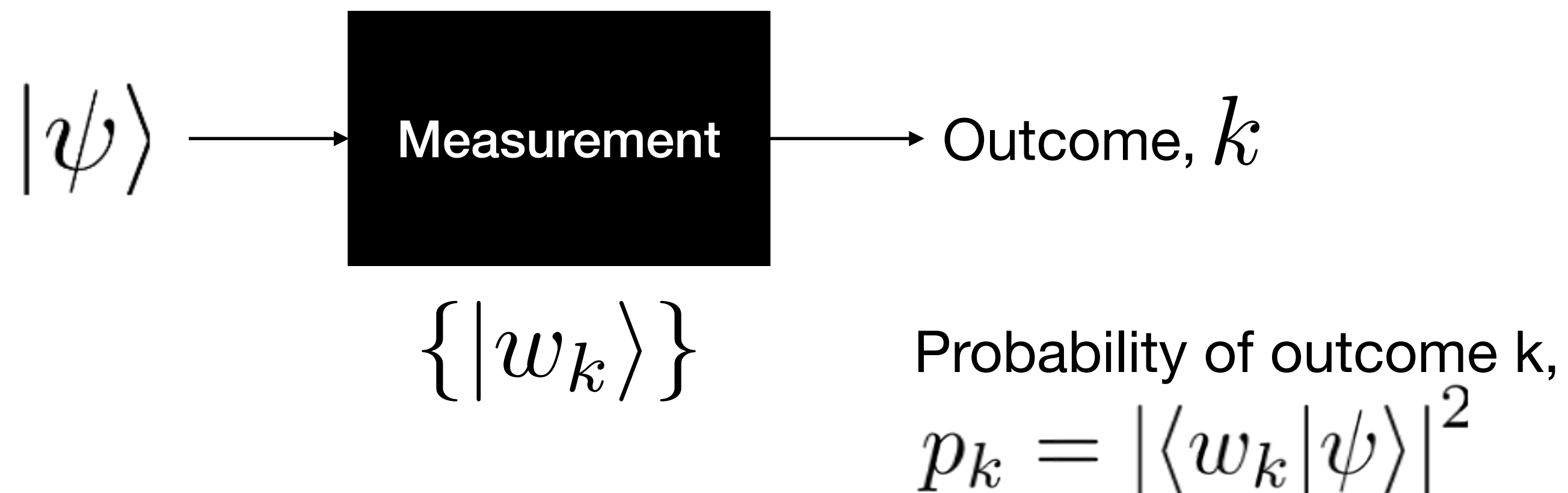
Transformations, Measurements and
Introduction to Applications

The Wasson Peak,
w/ Prof. Nicolas Cerf (U. Brussels)
October 2023



Quantum state and projective measurement

- Quantum (pure) state (of one or a collection of) mode(s): $|\psi\rangle$
- Projective measurement—
 - set of unit-norm orthonormal vectors: $\{|w_k\rangle\}$, $\langle w_k | w_j \rangle = \delta_{kj}$



Discriminating two states: Exercise 8

$|\psi_1\rangle$ (hypothesis H_1) vs. $|\psi_2\rangle$ (hypothesis H_2)

Prior probabilities: $p_1 = p$, $p_2 = 1 - p$

Inner product between
the two states

$$\langle\psi_1|\psi_2\rangle = \sigma = \cos 2\theta$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|w_1\rangle = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}$$

$$|\psi_1\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

$$|\psi_2\rangle = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}$$

$$|w_2\rangle = \begin{pmatrix} \sin \phi \\ -\cos \phi \end{pmatrix}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let us call the
measurement result,
 $z \in \{1, 2\}$

$$P(1|2) = |\langle w_1|\psi_2\rangle|^2 = (\cos \theta \cos \phi - \sin \theta \sin \phi)^2 = \cos^2(\theta + \phi)$$

$$P(2|1) = |\langle w_2|\psi_1\rangle|^2 = (\cos \theta \sin \phi - \sin \theta \cos \phi)^2 = \sin^2(\theta - \phi)$$

$$P_e = pP(2|1) + (1 - p)P(1|2)$$

Find the value of ϕ
that minimizes P_e

$$\frac{\partial P_e}{\partial \phi} = 0 \Rightarrow$$

Optimal ϕ satisfies:

$$\text{A. } \tan(2\phi) = \frac{\cot(2\theta)}{2p + 1}$$

$$\text{B. } \tan(2\phi) = \frac{\tan(2\theta)}{2p - 1}$$

$$\text{C. } \tan(2\phi) = \frac{\tan(2\theta)}{2p + 1}$$

$$\text{D. } \tan(2\phi) = \frac{\cot(2\theta)}{2p - 1}$$

Minimum probability of error (two states)

Helstrom Bound

- Substituting $\tan 2\phi = \frac{\tan 2\theta}{2p - 1}$ yields:

Probability of error $P_e = \frac{1}{2} \left[1 - \sqrt{1 - 4p(1 - p)\sigma^2} \right] := P[H_1|z = 2] = P[H_2|z = 1]$

Probability of correct decision $P_c = \frac{1}{2} \left[1 + \sqrt{1 - 4p(1 - p)\sigma^2} \right] := P[H_1|z = 1] = P[H_2|z = 2]$

For equal priors, i.e., $p = 1/2$, $P_{e,\min} = \frac{1}{2} \left[1 - \sqrt{1 - \sigma^2} \right]$

Number states basis for any pure state of a mode

- Number states $|n\rangle$ and $|m\rangle$ can be distinguished with zero error (photon detection)
- Hence they must be orthogonal states, $\sigma = \langle n|m\rangle = 0, \forall n \neq m$
- Number states (of a mode) span all possible energies a mode can have
- So, it is reasonable that they must form a basis to represent *any* state of a mode

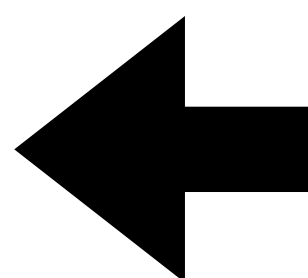
$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad \sum_{n=0}^{\infty} |c_n|^2 = 1$$
- Recall: coherent state written as $|\alpha\rangle = \sum_{n=0}^{\infty} \left(\frac{e^{-|\alpha|^2/2} \alpha^n}{\sqrt{n!}} \right) |n\rangle$
- Let us next calculate inner product of two coherent states, e.g., the OOK alphabet considered in Exercise 5

Inner product between two coherent states

Exercise 9

- Calculate the inner product between the two coherent states $|0\rangle$ and $|\alpha\rangle$
 - Assume, $\alpha \in \mathbb{R}$, and $N = \alpha^2$ the mean photon number in $|\alpha\rangle$
 - The inner product, $\sigma \equiv \langle 0|\alpha\rangle$ is given by:

- A. $\sigma = e^{-N}$

- B. $\sigma = e^{-N/2}$ 

- C. $\sigma = e^{-2N}$

- D. $\sigma = 0$

$$|\alpha\rangle = \sum_{n=0}^{\infty} \left(\frac{e^{-|\alpha|^2/2} \alpha^n}{\sqrt{n!}} \right) |n\rangle$$

Discriminating coherent states of light

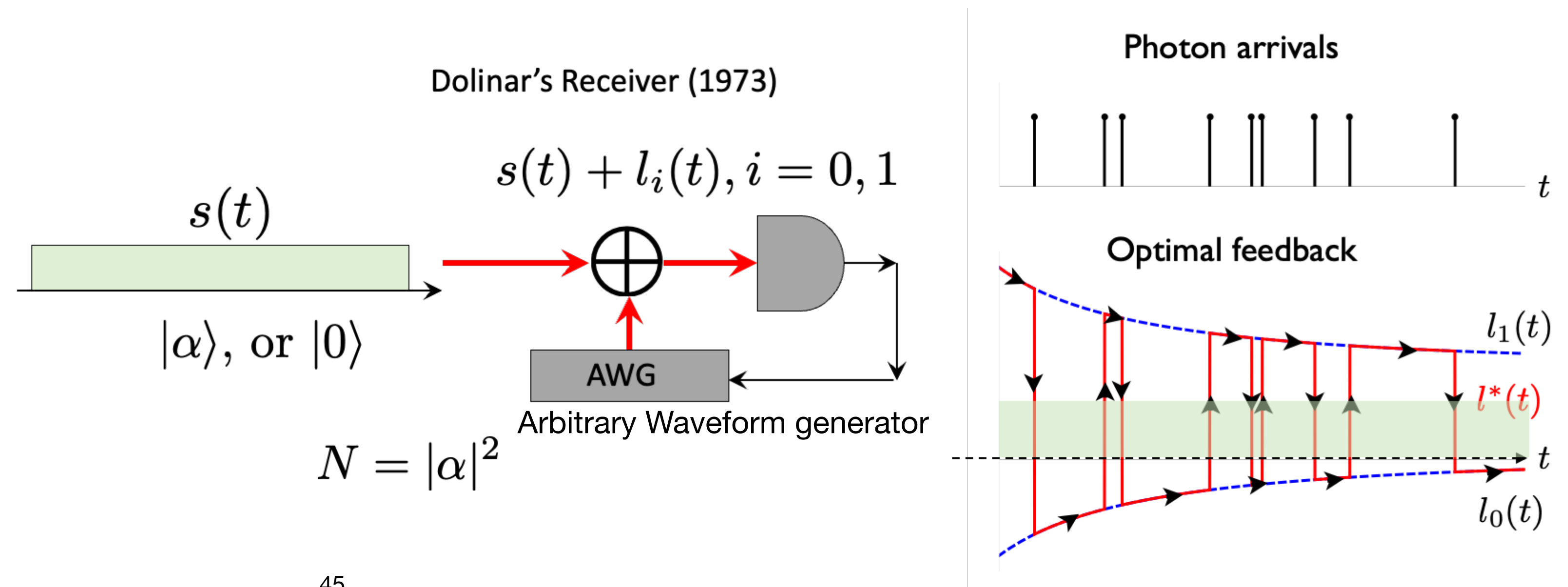
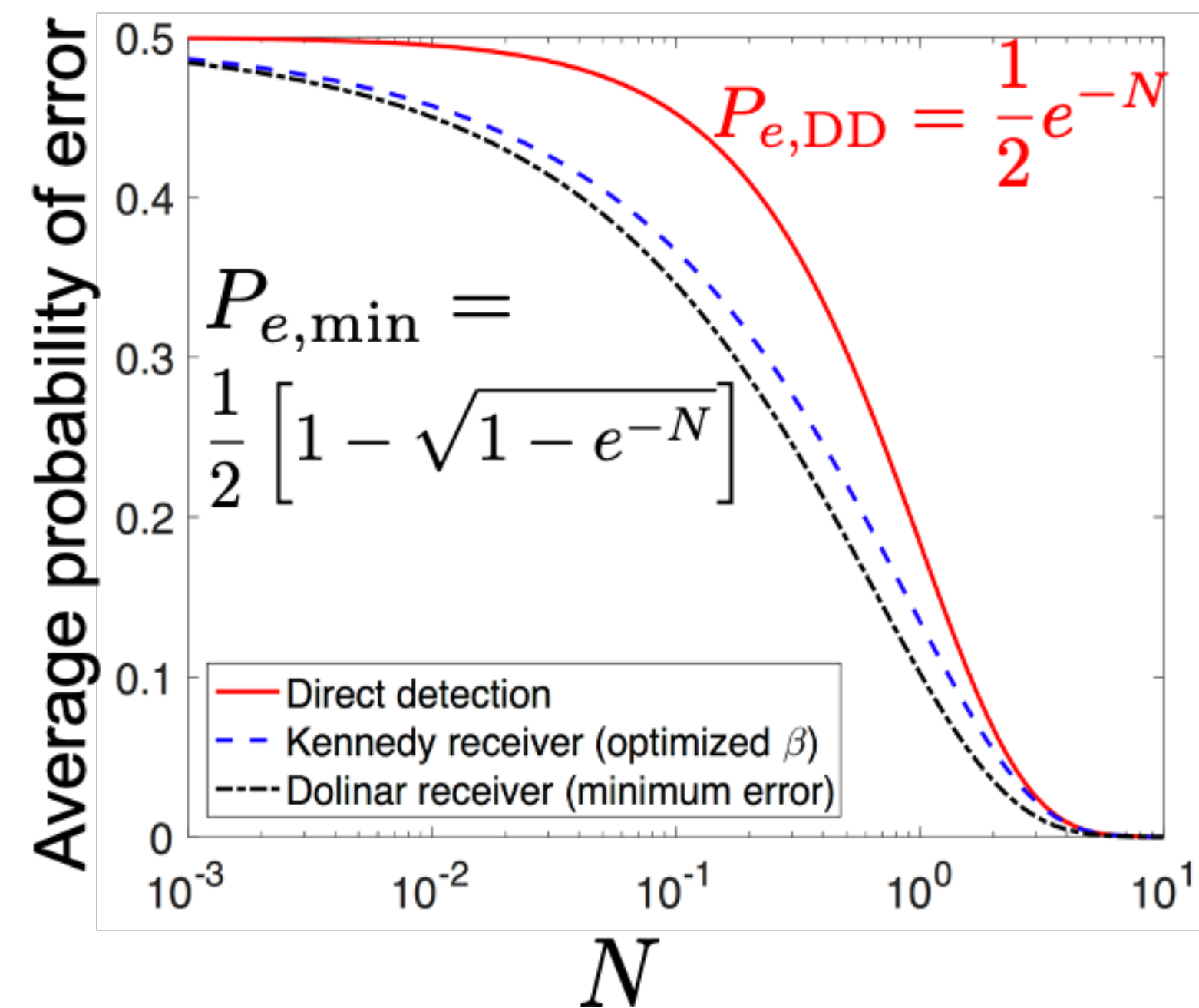
- Probability of error for telling apart the OOK coherent states $\{|0\rangle, |\alpha\rangle\}$, $N = |\alpha|^2$

Helstrom's Bound

$$P_{e,\min} = \frac{1}{2} \left[1 - \sqrt{1 - \sigma^2} \right]$$

$$\sigma = e^{-N/2}$$

- Ideal photon counting (Direct Detection): Exercise 5
- Displacement based (Kennedy's) receiver
- Helstrom bound, derived in Exercise 8, using inner product from Exercise 9



Discriminating coherent states

Even if excited in orthogonal modes, zero-error discrimination is not possible

- Inner product with any two coherent states, $\langle \alpha | \beta \rangle = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha^* \beta} \neq 0, \forall \alpha, \beta \in \mathbb{C}$
- So, two coherent states can never be told apart with zero probability of error
- Even if they are excited in orthogonal modes. Why?

$$\text{Hypothesis 1 : } |\psi_1\rangle = |0\rangle|\alpha\rangle$$

$$P[H_1] = p$$

$$\text{Hypothesis 2 : } |\psi_2\rangle = |\beta\rangle|0\rangle$$

$$P[H_2] = 1 - p$$

$$\sigma = \langle \psi_1 | \psi_2 \rangle = \langle 0 | \beta \rangle \langle \alpha | 0 \rangle = e^{-(|\beta|^2 + |\alpha|^2)/2} > 0$$

$$P_{e,\min} = \frac{1}{2} \left[1 - \sqrt{1 - 4p(1-p)\sigma^2} \right] > 0$$

Discriminating between M states

- M-ary ensemble (priors, states): $\{p_i, |\psi_i\rangle\} ; i = 1, 2, \dots, M$, $\sigma_{ij} = \langle \psi_i | \psi_j \rangle$
- M-ary projective measurement that attains the minimum error probability
- Measurement projectors $\{|w_j\rangle\} , j = 1, 2, \dots, M$
- Find relative orientation of measurement vectors with state vectors: $x_{ij} = \langle w_i | \psi_j \rangle$
- “YKL conditions” for minimizing average error probability:

$$\begin{aligned}
 (1) \quad & p_m x_{km} x_{mm}^* = p_k x_{kk} x_{mk}^* \\
 (2) \quad & \sum_{k=1}^M x_{kj} x_{ki}^* = \sigma_{ij} \\
 (3) \quad & \sum_{j=1}^M x_{jj} |w_j\rangle \langle \psi_j| \geq p_i |\psi_i\rangle \langle \psi_i|, \forall i
 \end{aligned}
 \left. \vphantom{\begin{aligned} (1) \\ (2) \\ (3) \end{aligned}} \right\} \begin{aligned} & \text{M(M+1)/2 non-} \\ & \text{linear simultaneous} \\ & \text{equations: solve for } x_{ij} \\ & P_{e,\min} = 1 - \sum_{i=1}^M p_i |x_{ii}|^2 \end{aligned}$$

(Check for uniqueness of solution)

Discriminating multiple coherent states

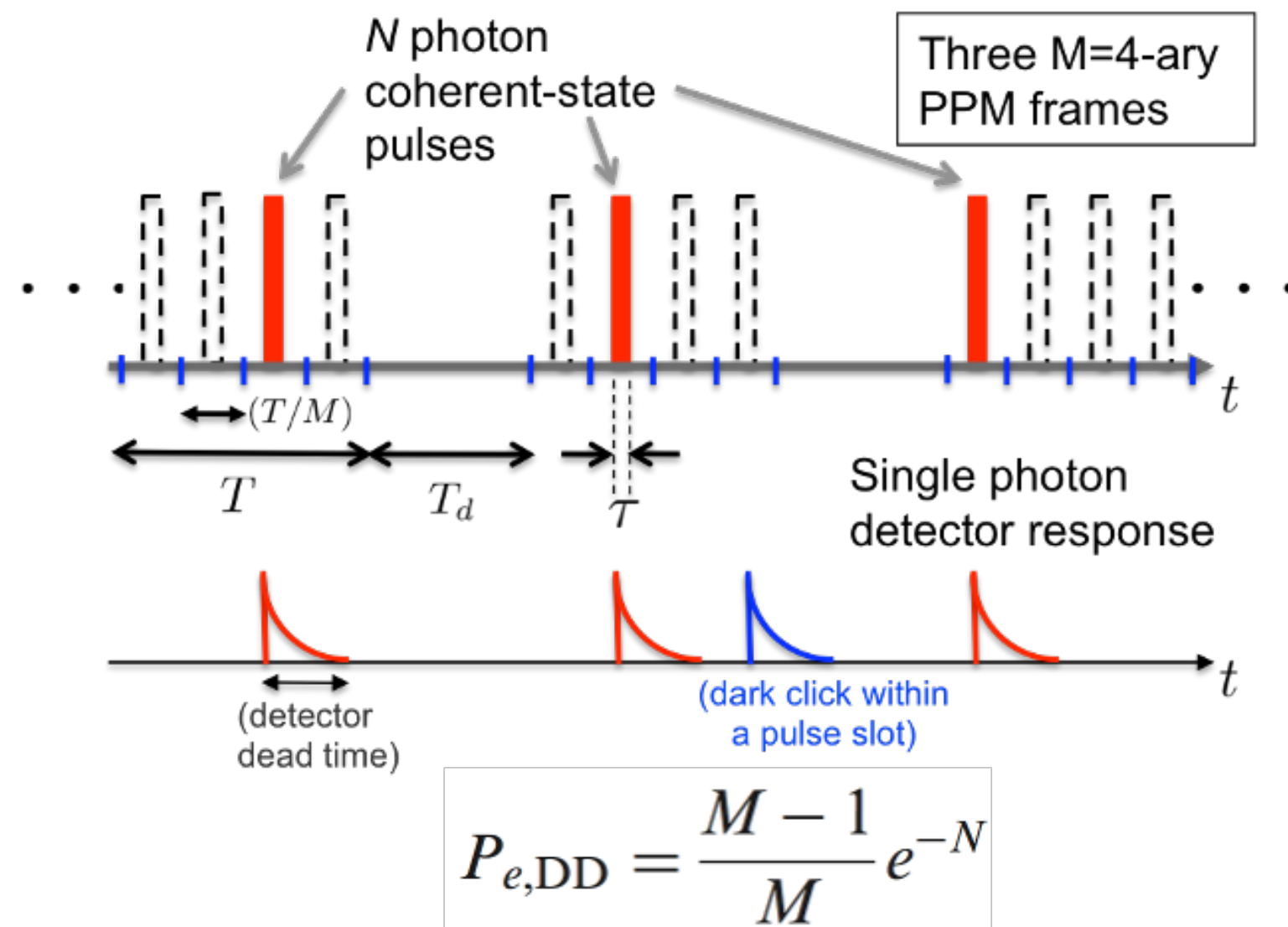
- **Homework:** Calculate the Helstrom bound for a simple set of three coherent states (i.e., a given mode has been excited in one of 3 coherent states, and the receiver must choose which one it is, such that the mean error probability is minimum)
 - Example 1 (BPSK-RZ): $\{ |-\alpha\rangle, |0\rangle, |\alpha\rangle \}$
 - Example 2 (QPSK): $\{ |\alpha\rangle, |i\alpha\rangle, |-\alpha\rangle, |-i\alpha\rangle \}$
 - Example 3 (PPM): $\{ |\alpha, 0, 0, 0\rangle, |0, \alpha, 0, 0\rangle, |0, 0, \alpha, 0\rangle, |0, 0, 0, \alpha\rangle \}$
- Receiver that attains the Helstrom bound must use *quantum* transformations of the coherent states prior to detection, e.g., “squeezing”, which is beyond the scope of this short course.

Conditional nulling receiver for Pulse Position Modulation

Direct detection

$$|\psi_i\rangle = |0\rangle \dots |0\rangle |\alpha\rangle |0\rangle \dots |0\rangle \quad |\alpha|^2 = N$$

1 ... i ... M



Quantum limit (YKL)

$$\sigma_{ij} = \langle \psi_i | \psi_j \rangle = |\langle \alpha | 0 \rangle|^2 = e^{-N}, i \neq j$$

$$x_{ij} = b, i \neq j; x_{ii} = a, \forall i$$

(symmetry postulate)

YKL conditions for minimum error probability

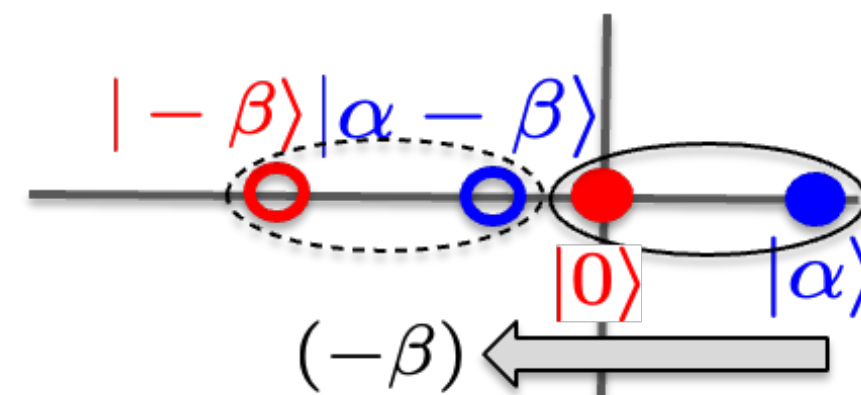
$$a^2 + (M-1)b^2 = 1$$

$$2ab + (M-2)b^2 = e^{-N}$$

$$P_{e,\min} = 1 - a^2$$

$$= \frac{M-1}{M^2} \left[\sqrt{1 + (M-1)e^{-N}} - \sqrt{1 - e^{-N}} \right]^2$$

$$\sim e^{-2N}, Me^{-N} \ll 1$$

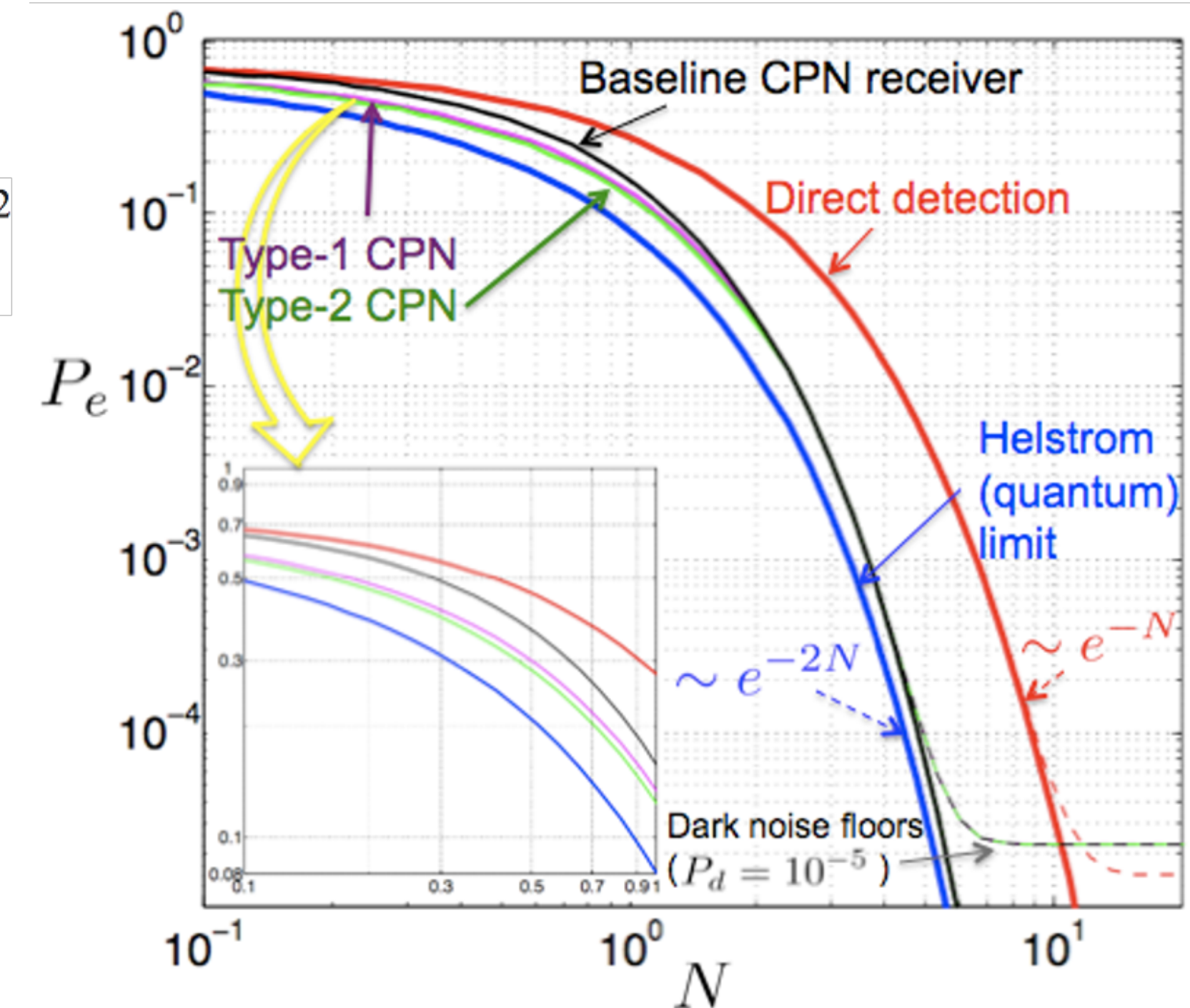


Conditional pulse nulling (CPN) receiver: uses beamsplitters, coherent states, photon detection, and classical feed-forward

Dolinar, MIT Ph.D. Thesis 1976, TDA Progress Report, 42-72, 1982

Guha, Habib, Takeoka, J. Mod. Optics, Vol. 58, 257-265, 2011

Chen, Habib, Dutton, Lazarus, Guha, Nature Photonics, 6, 374, 2012



Coherent state discrimination with a quantum computer

$$|\psi_1\rangle = |-\alpha\rangle|-\alpha\rangle|\alpha\rangle|-\alpha\rangle|\alpha\rangle$$

$$|\psi_2\rangle = |\alpha\rangle|\alpha\rangle|\alpha\rangle|\alpha\rangle|\alpha\rangle$$

$$|\psi_3\rangle = |-\alpha\rangle|-\alpha\rangle|\alpha\rangle|\alpha\rangle|-\alpha\rangle$$

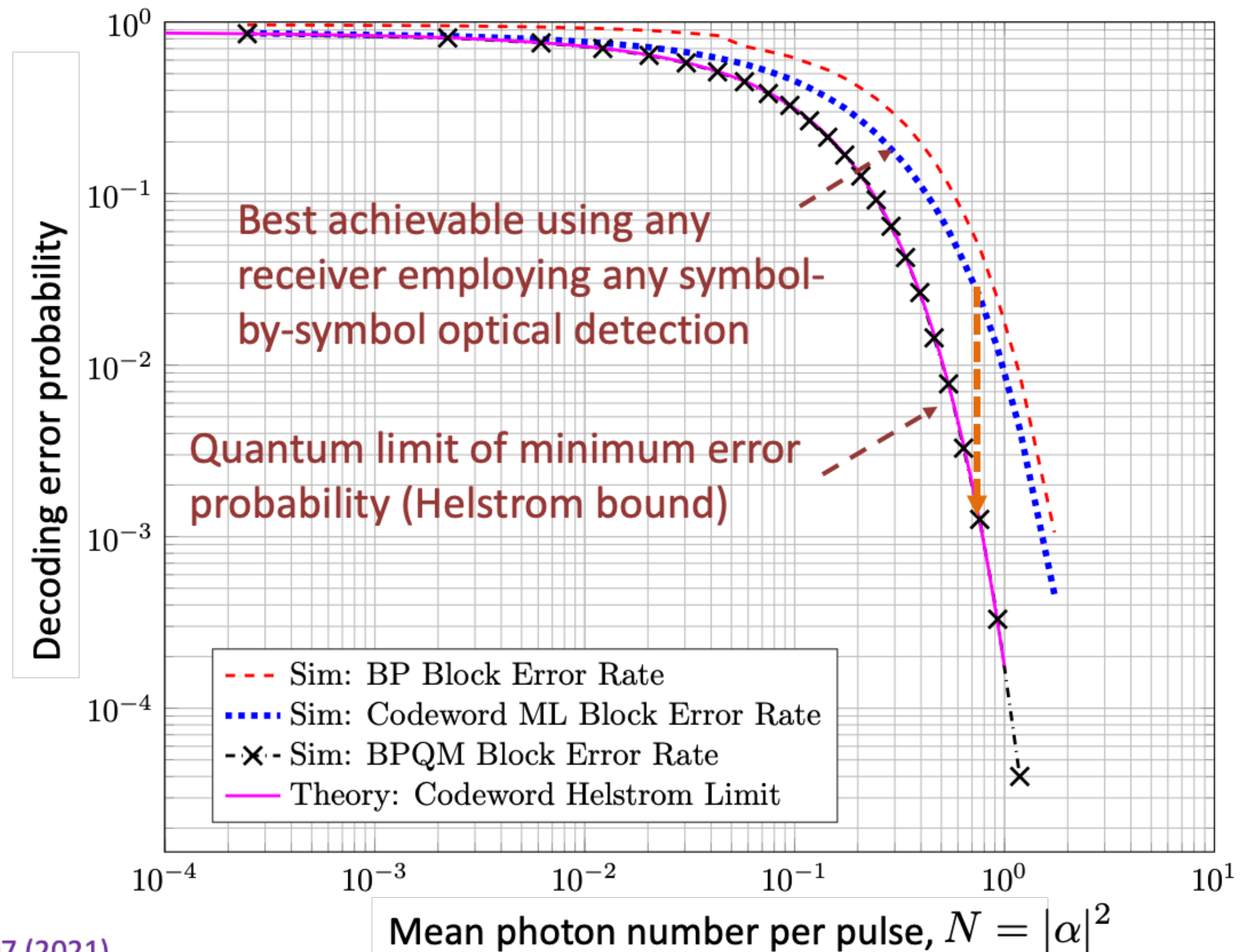
$$|\psi_4\rangle = |\alpha\rangle|\alpha\rangle|\alpha\rangle|-\alpha\rangle|-\alpha\rangle$$

$$|\psi_5\rangle = |\alpha\rangle|-\alpha\rangle|-\alpha\rangle|\alpha\rangle|\alpha\rangle$$

$$|\psi_6\rangle = |\alpha\rangle|-\alpha\rangle|-\alpha\rangle|-\alpha\rangle|-\alpha\rangle$$

$$|\psi_7\rangle = |-\alpha\rangle|\alpha\rangle|-\alpha\rangle|\alpha\rangle|-\alpha\rangle$$

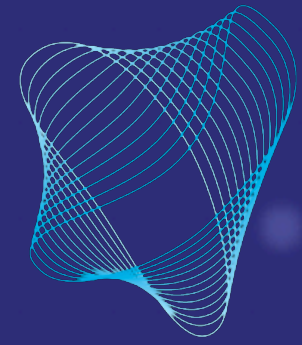
$$|\psi_8\rangle = |-\alpha\rangle|\alpha\rangle|-\alpha\rangle|-\alpha\rangle|\alpha\rangle$$



Photonic Quantum Information Processing

Outlook: quantum enhanced communications, sensing, computing

- Writing the state of the optical field in the quantum language allows us to evaluate “fundamental limits” (of precision) for information processing
- Finding structured designs of transmitters, modulation formats, receivers, etc. that meet the quantum limits is a hard task, with many problems still open
- Example applications of quantum states of light and quantum measurements
 1. Quantum receivers enhance capacity of deep space lasercom and reduces transmitter peak power
 2. Injection of squeezed light can enhance the sensitivity of radio-frequency photonics antennas
 3. Squeezed light enhances sensitivity of the Laser Interferometer Gravitational-Wave Observatory (LIGO)
 4. Photons enable generating entanglement among distant quantum memories
 5. Distributed entanglement can enhance the resolution of a collection of distant astronomical telescopes
- We hope this course will enthuse some of you to pursue a rigorous study of quantum optics, and quantum information theory!



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<https://cqn-erc.org/>

Course Evaluation Survey

We value your feedback on all aspects of this short course. Please go to the link provided in the Zoom Chat or in the email you will soon receive to give your opinions of what worked and what could be improved.

2026 CQN Winter School on Quantum Networks

Funded by National Science Foundation Grant #1941583

