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Quantum Networks
NSF Engineering Research Center

Theory of Quantum Channels for Photonic Quantum Networks

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INTRODUCTION

The Era of Quantum Engineering

LIGO

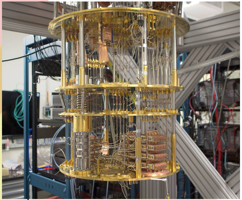
Quantum sensing

Using non-classical source to enhance hypothesis testing and parameter estimation

. . . . . .

As sub-routine in communication and computing

Quantum computer
IBM, >50 qubits,
Google supremacy,



Quantum communication

Quantum key distribution
Quantum teleportation
Quantum network
Entanglement-assisted communication

. . . . . .

Enabling non-classical resource at distance

Quantum satellite

Quantum computing

Quantum adiabatic algorithms
Quantum circuit model
Measurement-based quantum computing
NISQ quantum computation
Quantum machine learning

Enables receiver design,
Basic components

1

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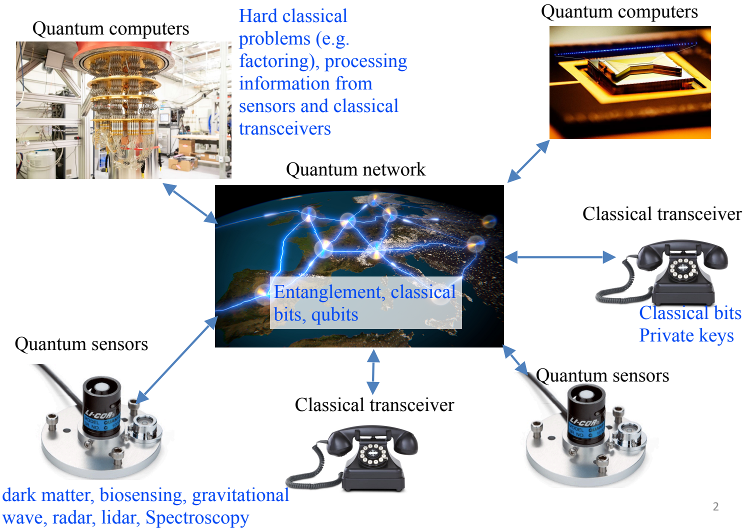
Quantum Networking



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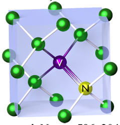
Many physical platforms and systems

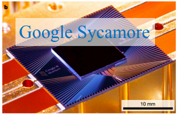
NV centers, color centers



Waldherr et al. Nature **506**, 204 (2014)

Long coherence time

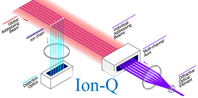
SC transmon qubits



Arute et al., Nature **574**, 505 (2019).

Large scale, strong interaction

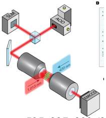
Trapped-ion



Wright et al., Nat. Commun. **10**, 5464 (2019)

High fidelity, high circuit volume

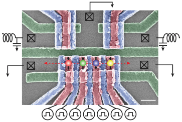
Rydberg atoms



Nature **595**, 227–232 (2021)

Large scale, high fidelity

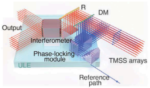
Electron spin in Q dots



Kandel et al., Nature **573** 553 (2019)

Long coherence time

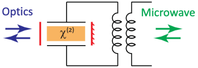
Bulk optics/nanophotonics



Zhong et al., Science **370** 1460 (2020)

Scalable, sensing/communication

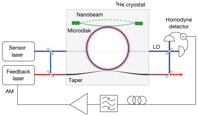
Transduction systems



Fan et al., Science Adv. **4** aar4994 (2018)

networking

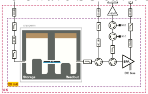
Optomechanical systems



Wilson et al., Nature **524** 325 (2015)

Force sensing

Microwave SC cavities



Campagne-Ibarcq et al., Nature **584**, 368 (2020)

Strong interaction, robustness

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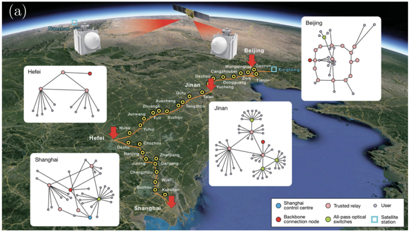
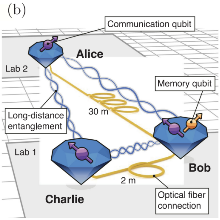




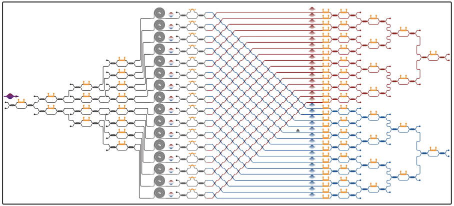


Fig. Top: (a) Secure quantum communication network [YA Chen et al 2020]. (b) Proof-of-principle quantum network [Pompili et al 2021]. Bottom: Schematic of photonic quantum circuit. [J Wang et al 2018]

Poll 0: Photonic quantum information processing

What are photons good for?

(A) Quantum sensing
(B) Quantum computing
(C) Quantum communication
(D) All of the above

Answer 0: Photonic quantum information processing

What are photons good for?

(A) Quantum sensing

(B) Quantum computing

(C) Quantum communication

(D) All of the above

Photons are quite versatile. Several quantum computing approaches exists that are based on linear optics, microwave cavity modes, etc. Quantum sensing platforms have been demonstrated with microwave cavities, quantum optical setups, etc. Finally, quantum information processing across long distances will require quantum optical interconnects.

QUANTUM CHANNELS: GENERAL DESCRIPTION

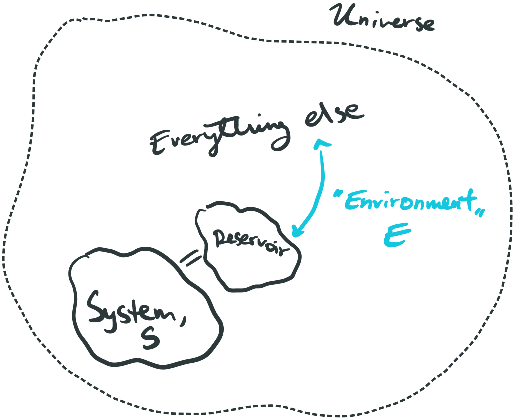
Notation

Symbol	Description
$\mathscr{H}$	Hilbert space
$\|\Psi\rangle, \rho$	Pure quantum state, density matrix
$\mathbb{I}$	Identity operator
$\hat{U}$	Unitary operator ($\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = \mathbb{I}$)
$\mathcal{N}, \mathcal{L}$ etc.	Quantum channel
$\hat{a}, \hat{a}^\dagger$	Creation, annihilation operators
$\hat{q}, \hat{p}$	Position, momentum operators
$\hat{H}$	Hamiltonian
$\|\text{vac}\rangle$	Vacuum quantum state
$\|0\rangle, \|1\rangle$	Computational state (0 or 1)

Fundamentals: Church of the Larger Hilbert Space

Tenets of quantum physics:

- Taken all together, our universe evolves under **unitary dynamics**
- If we include everything—system, environment etc.—the **global quantum state is always pure**.



Fundamentals: Church of the Larger Hilbert Space

Mixed state comes from ignoring environment. Mathematically described by a partial trace over environmental subsystem.

1. Joint state of system S and environment E is a pure state $|\Psi\rangle_{SE}$.
2. Partial trace over environment, $\rho_S = \mathrm{Tr}_E\, |\Psi\rangle\langle\Psi|_{SE}$. Using Schmidt decomposition, $|\Psi\rangle_{SE} = \sum_i \sqrt{\lambda_i}\, |i\rangle_S\, |i\rangle_E$,

$$\begin{aligned}
\rho_S &= \mathrm{Tr}_E(|\Psi\rangle\langle\Psi|_{SE}) \\
&= \sum_j \langle j|_E \underbrace{\left(\sum_i \sqrt{\lambda_i}\, |i\rangle_S\, |i\rangle_E\right)}_{|\Psi\rangle_{SE}} \underbrace{\left(\sum_k \sqrt{\lambda_k}\, \langle k|_S\, \langle k|_E\right)}_{\langle\Psi|_{SE}} |j\rangle_E \\
&= \sum_j \lambda_j\, |j\rangle\langle j|_S
\end{aligned}$$

where λ_j's are eigenvalues of ρ_S (i.e., "probability of outcome j")

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Fundamentals: Church of the Larger Hilbert Space

Invert the process (**purification**):

1. Start with mixed state of the system, $\rho_S = \sum_j \lambda_j \left|j\right\rangle\!\left\langle j\right|_S$, where λ_j's are eigenvalues of ρ_S
2. Append an environment E with basis $\left|j\right\rangle_E$ (free to choose basis as we like), so that $\left|\Psi\right\rangle_{SE} = \sum_j \sqrt{\lambda_j} \left|j\right\rangle_S \left|j\right\rangle_E$

* If necessary, we can continue this process until the entire state of the resulting ubersystem (the universe, SE) is a pure state.

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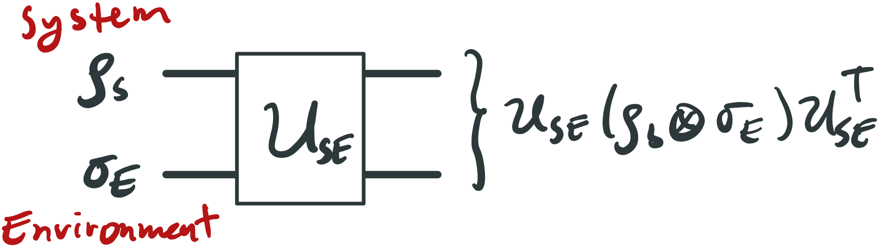
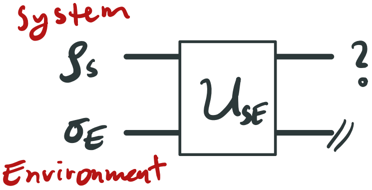


- Quite generally, evolution of a quantum system can be modeled by unitary evolution, where system S interacts with larger environment E



- What is the state of the system, by itself, after evolution?

$$\rho'_S = \mathrm{Tr}_E\, \hat{U}_{SE}\, (\rho_S \otimes \sigma_E)\, \hat{U}^\dagger_{SE}$$

1. Input-output map $\mathcal{N} : \rho_S \to \rho'_S$ is a **quantum channel**, $\mathcal{N}(\rho_S) = \rho'_S$
2. Observe all operations are linear, i.e. $\mathcal{N}(\rho + \tau) = \mathcal{N}(\rho) + \mathcal{N}(\tau)$

- In math language, a quantum channel is a completely positive trace-preserving (CPTP) map
- Example: Output state of photon after traversing a single-mode fiber.

Fundamentals: Quantum channels and Kraus operators

- Define quantum channel $\mathcal{N}$ from unitary evolution as

$$\mathcal{N}(\rho_S) := \mathrm{Tr}_E \left\{ \hat{U}_{SE} \left(\rho_S \otimes \sigma_E \right) \hat{U}_{SE}^{\dagger} \right\}$$

- To further simplify, take $\sigma_E = |e_0\rangle\langle e_0|_E$. [This is always possible by making environment large enough.] And pick basis $|e_k\rangle_E$. Then,

$$\begin{aligned} \mathcal{N}(\rho_S) &= \sum_k \langle e_k|_E \left(\hat{U}_{SE} \rho_S \otimes |e_0\rangle\langle e_0|_E \, \hat{U}_{SE}^{\dagger} \right) |e_k\rangle_E \\ &= \sum_k \left(\langle e_k|_E \, \hat{U}_{SE} \, |e_0\rangle_E \right) \rho_S \left(\langle e_0|_E \, \hat{U}_{SE}^{\dagger} \, |e_k\rangle_E \right) \\ &= \sum_k \hat{L}_k \rho_S \hat{L}_k^{\dagger} \quad \textbf{(Kraus operators)} \end{aligned}$$

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Fundamentals: Quantum channels and Kraus operators

- Channel evolution: $\mathcal{N}(\rho_S) = \sum_k \hat{L}_k \rho_S \hat{L}_k^\dagger$ with Kraus operators $\hat{L}_k \coloneqq \langle e_k|_E \hat{U}_{SE} |e_0\rangle_E$.
- Using completeness $\sum_k |e_k\rangle\langle e_k|_E = \mathbb{I}_E$ and definition of $\hat{L}_k$'s,

$$\begin{aligned}\sum_k \hat{L}_k^\dagger \hat{L}_k &= \sum_k \langle e_0|_E \hat{U}_{SE}^\dagger |e_k\rangle\langle e_k|_E \hat{U}_{SE} |e_0\rangle_E \\ &= \langle e_0| \overbrace{\mathbb{I}_S \otimes \mathbb{I}_E}^{\hat{U}_{SE}^\dagger \hat{U}_{SE} =} |e_0\rangle \\ &= \mathbb{I}_S.\end{aligned}$$

- These relations ensure $\mathcal{N}(\rho_S)$ is a good quantum state—e.g., $\mathrm{Tr}\{\mathcal{N}(\rho_S)\} = 1$, $(\mathcal{N}(\rho_S))^\dagger = \mathcal{N}(\rho_S)$ etc.

* Note: Given $\mathcal{N}_1$ and $\mathcal{N}_2$, concatenation $\mathcal{N}_2 \circ \mathcal{N}_1$ or $\mathcal{N}_1 \circ \mathcal{N}_2$ is also a quantum channel. [Supplementary exercise for the student]

Scratch paper

Fundamentals: Real-world quantum channels

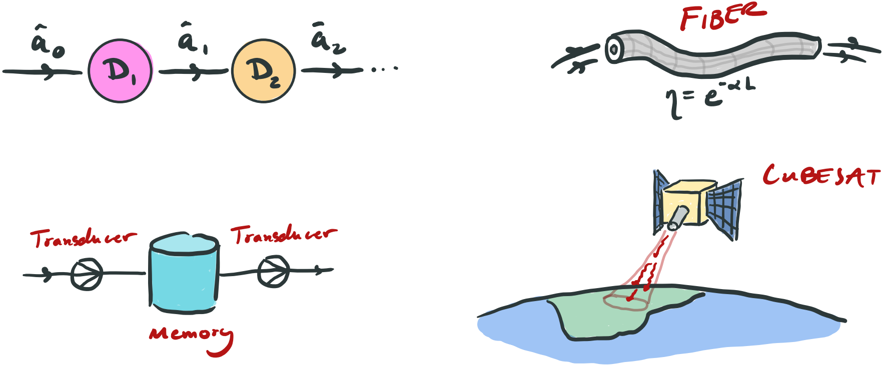


Figure: Cascaded quantum systems, fiber transmission, quantum interconnects, free-space evolution.

Examples: Erasure and Depolarizing

$\mathcal{L}_\varepsilon$: Consider a two-level quantum system (a qubit) described by the quantum state $\Psi \in \mathscr{H}$, and consider the "erasure state" $|\varepsilon\rangle$ which lies outside of $\mathscr{H}$ (i.e., $\langle\varepsilon|\Psi|\varepsilon\rangle = 0 \,\forall\, \Psi \in \mathscr{H}$). An **erasure channel** $\mathcal{L}_\varepsilon$ acts on the qubit as,

$$\mathcal{L}_\varepsilon(\Psi) = (1-\varepsilon)\Psi + \varepsilon\,|\varepsilon\rangle\!\langle\varepsilon|\,,$$

where $0 \leq \varepsilon \leq 1$ is the erasure probability. For single-photon qubit encodings, photon loss can be modeled as an erasure channel.

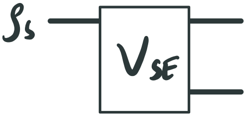
Δ_p: Given a qubit Ψ, a **depolarizing channel** Δ_p acts as follows,

$$\Delta_p(\Psi) = (1-p)\Psi + p\hat{I}/2,$$

where $\hat{I}/2$ is the maximally mixed state and $0 \leq p \leq 4/3$. Depolarizing noise is a common, simplified noise model used to assess fault-tolerance of quantum computing architectures.

Fundamentals: Unitary & Isometric Extensions

- Often useful to have different, though equivalent, descriptions of quantum channels.
- **Unitary extension**: $\mathcal{N}(\rho_S) = \mathrm{Tr}_E\left\{\hat{U}_{SE}\left(\rho_S \otimes \sigma_E\right)\hat{U}_{SE}^{\dagger}\right\}$
  - → Unitary because $\hat{U}_{SE}^{\dagger}\hat{U}_{SE} = \hat{U}_{SE}\hat{U}_{SE}^{\dagger} = \mathbb{I}_{SE}$
- **Isometric extension** (lazy version of unitary extension): $\mathcal{N}(\rho_S) = \mathrm{Tr}_E\left\{\hat{V}_{SE}\rho_S\hat{V}_{SE}^{\dagger}\right\}$ where $\hat{V}_{SE} = \sum_k \hat{L}_k \otimes |e_k\rangle_E$.
  - → Isometric because $\hat{V}_{SE}^{\dagger}\hat{V}_{SE} = \mathbb{I}_S$ but $\hat{V}_{SE}\hat{V}_{SE}^{\dagger} = \Pi_{SE}$ where Π_{SE} is a projection, *not* the identity!



Fundamentals: Unitary & Isometric Extensions

- Why are extensions and purifications useful? → Theoretical tool for QIP analyses

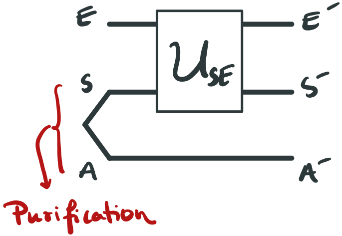


Figure: Extension and purification. Full system ASE and $A'S'E'$ are pure quantum states.

- E.g., Quantum capacity (single letter)

$$Q^{(1)}(\mathcal{N}) = \max_{\rho_S} S(\mathcal{N}(\rho_S)) - S(\mathcal{N}^c(\rho_S))$$

where $S(\cdot)$ is entropy and $S(\mathcal{N}^c(\rho_S)) = S(\rho_{E'}) = S(\rho_{A'S'})$.

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Poll 1: Concatenated erasures

Consider two erasure channels $\mathcal{L}_{\varepsilon_1}$ and $\mathcal{L}_{\varepsilon_2}$ where, e.g., $\mathcal{L}_{\varepsilon}(\Psi) = (1-\varepsilon)\Psi + \varepsilon\, |\varepsilon\rangle\langle\varepsilon|$ for some state Ψ. The concatenation of the two erasure channels is another erasure channel, $\mathcal{L}_{\varepsilon_{12}} = \mathcal{L}_{\varepsilon_2} \circ \mathcal{L}_{\varepsilon_1}$. What is the erasure probability ε_{12}? [Hint: The erasure probability is 1 minus the transmission probability.]

(A) $(\varepsilon_1 + \varepsilon_2)/2$

(B) $\varepsilon_1 \varepsilon_2$

(C) $1 - (1-\varepsilon_1)(1-\varepsilon_2)$

Answer 1: Concatenated erasures

Consider two erasure channels $\mathcal{L}_{\varepsilon_1}$ and $\mathcal{L}_{\varepsilon_2}$ where, e.g., $\mathcal{L}_{\varepsilon}(\Psi) = (1-\varepsilon)\Psi + \varepsilon\,|\varepsilon\rangle\langle\varepsilon|$ for some state Ψ. The concatenation of the two erasure channels is another erasure channel, $\mathcal{L}_{\varepsilon_{12}} = \mathcal{L}_{\varepsilon_2} \circ \mathcal{L}_{\varepsilon_1}$. What is the erasure probability ε_{12}? [Hint: The erasure probability is 1 minus the transmission probability.]

(A) $(\varepsilon_1 + \varepsilon_2)/2$

(B) $\varepsilon_1 \varepsilon_2$

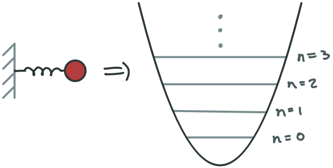
(C) $1 - (1-\varepsilon_1)(1-\varepsilon_2)$

State either gets transmitted or erased. Transmission probability for first channel is $(1-\varepsilon_1)$. Transmission probability for second channel is $(1-\varepsilon_2)$. Total transmission probability is the product of probabilities $(1-\varepsilon_1)(1-\varepsilon_2)$. Erasure probability is thus $1-(1-\varepsilon_1)(1-\varepsilon_2)$.

GAUSSIAN BOSONIC CHANNELS

Crash course: Quantum harmonic oscillator

- Free EM field is bosonic field described by harmonic-oscillator–like Hamiltonian $\hat{H}_{\text{osc}} = \frac{\hbar\omega}{2}\left(\hat{q}^2 + \hat{p}^2\right)$ with frequency ω



- Canonical **position** $\hat{q}$ and **momentum** $\hat{p}$ (**quadratures**) obey CCR $[\hat{q}, \hat{p}] = i\hat{I}$
- Annihilation operator $\hat{a}$ related via $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$ s.t. $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{I}$

- Equivalently, $\hat{H}_{\text{osc}} = \hbar\omega\hat{n} + \hbar\omega/2$ with number operator $\hat{n} \equiv \hat{a}^\dagger\hat{a}$ and eigenstate (**number state**) $|n\rangle$ where $n \in \mathbb{Z}^+$.
- Explicitly, $|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n}{\sqrt{n!}}|\text{vac}\rangle$. Then, $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. Note $\hat{a}|\text{vac}\rangle = 0$.

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Gaussian unitaries: Some basics

- We will follow the previous approach: Start from a unitary, then "trace out environment" to define bosonic (Gaussian) channels
- First, notation. Vector of position/momentum operators for N modes

$$\hat{\boldsymbol{x}} := (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_N, \hat{p}_N)$$

which obey canonical commutation relations $[\hat{\boldsymbol{x}}_n, \hat{\boldsymbol{x}}_m] = i\boldsymbol{\Omega}_{nm}$ with $\boldsymbol{\Omega} = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. [Diagonal blocks of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, everywhere else 0.]

- Unitary $\hat{U}(\lambda) = e^{-i\hat{H}\lambda}$ is generated from $\hat{H}$
  - $\rightarrow$ **Gaussian unitary**: $\hat{H}$ second order in $\hat{q}$'s and $\hat{p}$'s, e.g. $\hat{q}^2, \hat{q}\hat{p}$ etc.
  - $\rightarrow$ **Non-Gaussian unitary**: Higher order in $\hat{q}$'s and $\hat{p}$'s, e.g. $\hat{q}^3$

* Gaussian unitaries result in **linear evolution**, $\hat{U}_{\mathcal{G}}^{\dagger}\hat{\boldsymbol{x}}\hat{U}_{\mathcal{G}} = \boldsymbol{S}\hat{\boldsymbol{x}} + \boldsymbol{d}$. The (real, symplectic) matrix $\boldsymbol{S}$ and (real) vector $\boldsymbol{d}$ can be related to $\hat{H}$.

  - Linear transformation above can be proved via Hadamard's lemma $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \ldots$.

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Scratch paper

Gaussian unitaries: Displacements

- Consider a linear Hamiltonian (in creation/annihilation variables)

$$\hat{H} = i(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \sqrt{2}(\mathrm{Re}\{\alpha\}\hat{p} - \mathrm{Im}\{\alpha\}\hat{q})$$

  with unitary **displacement operator** $\hat{D}(\alpha) := e^{i\hat{H}}$.

- Linear Hamiltonians:
  i. Generate displacements $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha$
  ii. Produces coherent (e.g., laser) state from vacuum, $\hat{D}(\alpha)\,|\mathrm{vac}\rangle =: |\alpha\rangle$.
  [Physically, classical current coupled to EM field to generate $|\alpha\rangle$.]
  iii. Multi-mode version $\hat{D}^\dagger(\boldsymbol{\xi})\hat{\boldsymbol{x}}\hat{D}(\boldsymbol{\xi}) = \hat{\boldsymbol{x}} + \boldsymbol{\xi}$
  iv. Form an operator basis (orthogonal, complete) via
  $\mathrm{Tr}\left\{\hat{D}(\boldsymbol{\xi})\hat{D}(\boldsymbol{\xi}')\right\} = \pi^N\delta(\boldsymbol{\xi} + \boldsymbol{\xi}')$, analogous to Pauli operators

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- Using $\mathrm{Tr}\left\{\hat{D}(\boldsymbol{\xi})\hat{D}(\boldsymbol{\xi}')\right\} = \pi^N \delta(\boldsymbol{\xi}+\boldsymbol{\xi}')$, define **characteristic function** for any (bounded) operator $\hat{A}$,

$$\chi(\boldsymbol{\xi};\hat{A}) := \mathrm{Tr}\left\{\hat{A}\hat{D}(\boldsymbol{\xi})\right\}, \quad \text{such that} \quad \hat{A} = \int_{\mathbb{R}^{2N}} \frac{\mathrm{d}\boldsymbol{\xi}}{\pi^N} \chi(\boldsymbol{\xi};\hat{A})\hat{D}^\dagger(\boldsymbol{\xi})$$

- Fourier transform to **Wigner function**,

$$W(\boldsymbol{x};\hat{A}) = \int \frac{\mathrm{d}^{2N}\boldsymbol{\xi}}{(2\pi)^{2N}} \exp\left(-i\boldsymbol{x}^\top \boldsymbol{\Omega}\boldsymbol{\xi}\right) \chi(\boldsymbol{\xi};\hat{A})$$

* Often useful because one can define (arbitrary) quantum state and speak in terms of a function (characteristic, Wigner) rather than density matrix, ρ. Practically can do **quantum state tomography** to map the Wigner function in phase space and characterize ρ.

Gaussian unitaries: Characteristic & Wigner function

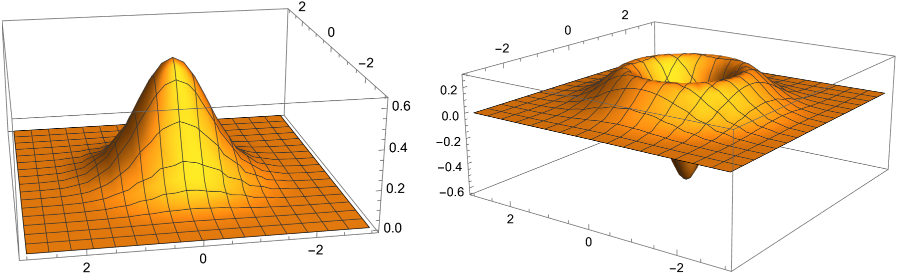


Figure: Example Wigner functions for (left) the vacuum state $|\mathrm{vac}\rangle$ and (right) single-photon state $|1\rangle$. Gaussian states, such as the vacuum, have positive Wigner functions. Negativity of the Wigner function, such as for a single-photon state, implies non-Gaussianity.

Gaussian unitaries: Characteristic & Wigner functions

- Characteristic function $\chi(\boldsymbol{\xi}; \hat{A})$ and Wigner function $W(\boldsymbol{x}; \hat{A})$ transform simply under Gaussian unitary evolution
- Recall $\hat{U}_{\mathcal{G}}^{\dagger} \hat{\boldsymbol{x}} \hat{U}_{\mathcal{G}} = \boldsymbol{S}\hat{\boldsymbol{x}} + \boldsymbol{d}$ for Gaussian unitary $\hat{U}_{\mathcal{G}}$
- Characteristic function: $\chi\left(\boldsymbol{\xi}; \hat{U}_{\mathcal{G}}^{\dagger} \hat{A} \hat{U}_{\mathcal{G}}\right) = \chi\left(\boldsymbol{S}^{-1}\boldsymbol{\xi}; \hat{A}\right) e^{i\boldsymbol{d}^{\top}\boldsymbol{\Omega}\boldsymbol{\xi}}$
- Wigner function: $W\left(\boldsymbol{x}; \hat{U}_{\mathcal{G}}^{\dagger} \hat{A} \hat{U}_{\mathcal{G}}\right) = W\left(\boldsymbol{S}^{-1}(\boldsymbol{\xi} - \boldsymbol{d}); \hat{A}\right)$
- The interpretation is that Gaussian unitaries $\implies$ linear transformations in phase space

Gaussian unitaries: Phase rotation

- Hamiltonian $\hat{H} = \hat{a}^\dagger \hat{a}$ generates **rotation** $e^{i\phi\hat{a}^\dagger\hat{a}}\hat{a}e^{-i\phi\hat{a}^\dagger\hat{a}} = e^{-i\phi}\hat{a}$
- Models free propagation, e.g. $\phi = kL$ where k is wavenumber and L is propagation length

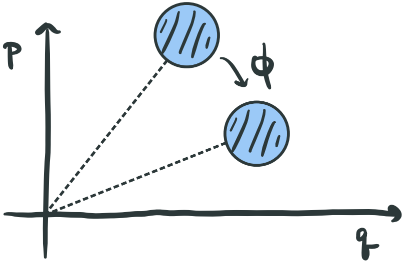


Figure: Wigner-function picture of phase rotation

Gaussian unitaries: Single-mode squeezing

- Hamiltonian $\hat{H} = \frac{r}{2i}(\hat{a}^2 - \hat{a}^{\dagger\,2}) = r\hat{q}\hat{p}$
- **Single-mode squeezing**: $\hat{S}(r)^\dagger \hat{q} \hat{S}(r) = e^{-r}\hat{q}$, $\hat{S}(r)^\dagger \hat{p} \hat{S}(r) = e^{r}\hat{p}$

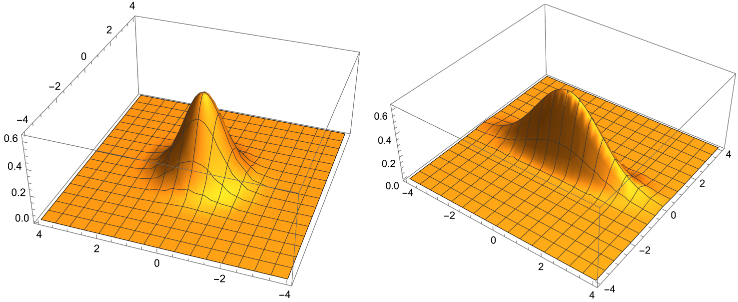


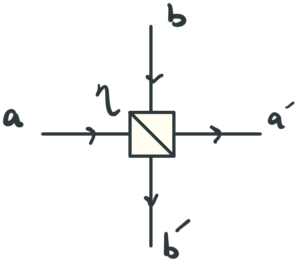
Figure: Squeezing the vacuum (left) generates squeezed light (right).

Gaussian unitaries: Beamsplitter

- Hamiltonian $\hat{H} \propto \hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}$
- **Beamsplitter** transformation:

$$\hat{B}(\theta)^\dagger \hat{a} \hat{B}(\theta) = \cos(\theta)\hat{a} + \sin(\theta)\hat{b}$$
$$\hat{B}(\theta)^\dagger \hat{b} \hat{B}(\theta) = \cos(\theta)\hat{b} - \sin(\theta)\hat{a}$$

  with transmission probability $\eta := \cos^2(\theta)$
- Coherent exchange of quanta (e.g., photons) between modes



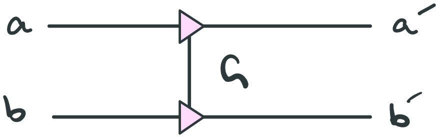
Gaussian unitaries: Two-mode squeezing

- Hamiltonian $\hat{H} \propto \hat{a}\hat{b} - \hat{a}^\dagger \hat{b}^\dagger$
- **Two-mode squeezing** transformation:

$$\hat{T}(r)^\dagger \hat{a} \hat{T}(r) = \cosh(r)\hat{a} + \sinh(r)\hat{b}^\dagger$$
$$\hat{T}(r)^\dagger \hat{b} \hat{T}(r) = \cosh(r)\hat{b} + \sinh(r)\hat{a}^\dagger$$

with gain $G := \cosh^2(r)$

- Particle pair creation; generates two-mode squeezed vacuum state



Non-Gaussian unitaries

- Based on higher-order Hamiltonians; some non-linearity required (e.g., enduced by qubit coupled to a photonic mode)
- Examples include (1) cubic phase gate, $\hat{H} = \hat{q}^3$ and (2) Kerr nonlinearity, $\hat{H} = \hat{n}^2$

* **Non-Gaussian elements (either non-Gaussian states and/or non-Gaussian operations) required for quantum error correction, fault-tolerant quantum computing etc.**

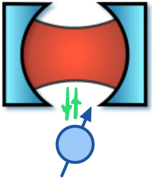
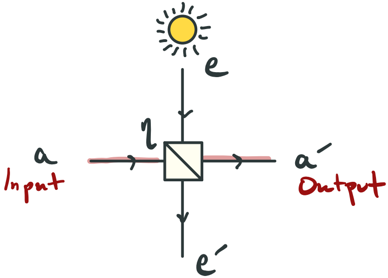


Figure: Two-level system (e.g., transmon qubit) coupled to a mode (e.g., microwave resonator) generates non-Gaussian transformations.

Gaussian channels: Thermal-loss channel

- Unitary channel discussed previously; non-unitary Gaussian channels lead to noise
- **Thermal loss channel** $\mathcal{L}_{\eta,N_E}$: Thermal environment interacting with mode on a beamsplitter. Models light propagation in fiber, free space, damped resonator mode etc.
- Transformation: $\hat{a}' = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e}$ and $\left\langle \hat{e}^\dagger \hat{e} \right\rangle = N_E$



Poll 2: Pure loss and erasure

A pure-loss channel $\mathcal{L}_\eta$ $(N_E = 0)$ has an operator sum representation

$$\mathcal{L}_\eta(\rho) = \sum_{\ell=0}^{\infty} \hat{A}_\ell \rho \hat{A}_\ell^\dagger, \tag{1}$$

with Kraus operators

$$\hat{A}_\ell = \sqrt{\frac{(1-\eta)^\ell}{\ell!}} \eta^{\hat{a}^\dagger \hat{a}/2} \hat{a}^\ell. \tag{2}$$

How many Kraus operators do we need to describe the output $\mathcal{L}_\eta(\rho_1)$ for a single-photon input state ρ_1? [Hint: focus on the $\hat{a}^\ell$ term and recall that $\hat{a}$ annihilates the vacuum.]

(A) 1 (B) 2 (C) 3

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Answer 2: Pure loss and erasure

A pure-loss channel $\mathcal{L}_\eta$ has an operator sum representation

$$\mathcal{L}_\eta(\rho) = \sum_{\ell=0}^{\infty} \hat{A}_\ell \rho \hat{A}_\ell^\dagger, \tag{3}$$

with Kraus operators

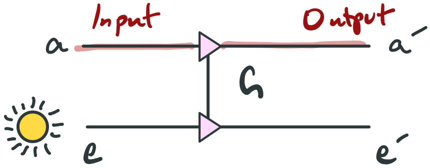
$$\hat{A}_\ell = \sqrt{\frac{(1-\eta)^\ell}{\ell!}} \eta^{\hat{a}^\dagger \hat{a}/2} \hat{a}^\ell. \tag{4}$$

How many Kraus operators do we need to describe the output $\mathcal{L}_\eta(\rho_1)$ for a single-photon input state ρ_1? [Hint: Focus on the $\hat{a}^\ell$ term and recall that $\hat{a}$ annihilates the vacuum.]

(A) 1 (B) 2 (C) 3

For single-photon state ρ_1 and $\ell \geq 2$, $\hat{a}^\ell \rho_1 \hat{a}^{\ell\,\dagger} = 0$ because $\hat{a}\rho_1\hat{a}^\dagger \propto |\mathrm{vac}\rangle\langle\mathrm{vac}|$ and $\hat{a}\,|\mathrm{vac}\rangle\langle\mathrm{vac}|\,\hat{a}^\dagger = 0$. Thus only first two Kraus operators ($\ell = 0, 1$) are necessary (and given by $\hat{A}_0 = \sqrt{\eta}\hat{I}$ and $\hat{A}_1 = \sqrt{1-\eta}\hat{a}$).

- **Thermal-amplifier channel** $\mathcal{A}_{G,N_E}$: Thermal environment interacting with mode via tow-mode squeezing. Models linear, phase-insensitive amplifiers
- Transformation: $\hat{a}' = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}^\dagger$ and $\langle \hat{e}^\dagger \hat{e} \rangle = N_E$



Gaussian channels: Additive Gaussian noise channel

- **Additive Gaussian noise channel** $\mathcal{N}_{N_E} = \lim_{\eta\to 1} \mathcal{L}_{\eta, \frac{N_E}{1-\eta}}$: Thermal noise added to the output. Models random walk in phase space, fluctuations of classical drive field etc.
- Transformation: $\hat{a}' = \hat{a} + \xi$ and $\overline{\xi^* \xi} = N_E$

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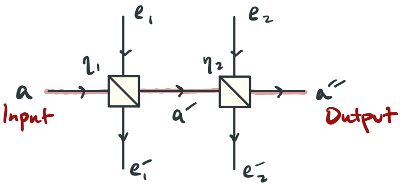
Scratch paper

Gaussian channels: Cascaded quantum channels



- Cascaded lossy devices modeled by thermal loss channels $\mathcal{L}_{\eta_i,N_i}$.
- Concatenated channel $\mathcal{L}_{\eta_2,N_2} \circ \mathcal{L}_{\eta_1,N_1}$ is a thermal loss channel $\mathcal{L}_{\eta_{12},N_{12}}$ with

$$\eta_{12} := \eta_1\eta_2 \quad \text{and} \quad N_{12} := \frac{\eta_2(1-\eta_2)N_1 + (1-\eta_2)N_2}{1-\eta_1\eta_2}$$

- Derivation:

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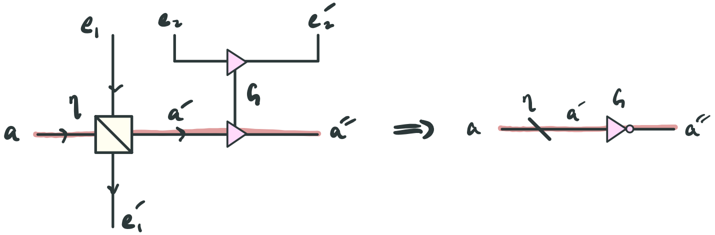
- Amplification $\mathcal{A}_{G,N_2}$ following a lossy device $\mathcal{L}_{\eta,N_1}$
- Resulting channel $\mathcal{A}_{G,N_2} \circ \mathcal{L}_{\eta,N_1}$ can be several things

$$\begin{cases} \mathcal{L}_{G\eta,N_3} \quad \text{with} \quad N_3 = \frac{G(1-\eta)}{1-G\eta} N_1 + \frac{G-1}{1-G\eta}(N_2+1), & G\eta < 1 \\ \mathcal{N}_{N_E} \quad \text{with} \quad N_E = (G-1)(N_1+N_2+1), & G\eta = 1 \\ \mathcal{A}_{G\eta,N_3'} \quad \text{with} \quad N_3' = \frac{G(1-\eta)}{G\eta-1}(N_1+1) + \frac{G-1}{G\eta-1} N_2, & G\eta > 1 \end{cases}$$



Exercise 1: Amplifier-then-loss is less noisy

Q: From before, we have that $\mathcal{N}_{N_{B_1}} = \mathcal{A}_{G,N_2} \circ \mathcal{L}_{\eta,N_1}$ for $G\eta = 1$, where $N_{B_1} = (G-1)(N_1 + N_2 + 1)$. Show that $\mathcal{N}_{N_{B_2}} = \mathcal{L}_{\eta,N_1} \circ \mathcal{A}_{G,N_2}$ for $G\eta = 1$ and give N_{B_2} explicitly. Prove that $N_{B_2} < N_{B_1}$. Hence, amp-loss is less noisy than loss-amp.

A: Use similar tricks and prove at level of annihilation operators.

$$\hat{a} \xrightarrow{\mathcal{A}_{G,N_2}} \hat{a}' = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}_2^\dagger$$

$$\hat{a}' \xrightarrow{\mathcal{L}_{\eta,N_1}} \hat{a}'' = \sqrt{\eta}\hat{a}' + \sqrt{1-\eta}\hat{e}_1.$$

Then $\hat{a}'' = \sqrt{\eta G}\hat{a} + \sqrt{1-\eta G}\left(\frac{\sqrt{\eta(G-1)}\hat{e}_2^\dagger + \sqrt{1-\eta}\hat{e}_1}{\sqrt{1-\eta G}}\right)$. Equivalent to AGN $\mathcal{N}_{B_2}$ in limit $\eta G \to 1$ with $N_{B_2} = (1-\eta)(N_2 + N_1 + 1)$. Since $1 - \eta = (G-1)/G$ and $(G-1)/G < G - 1$, then $N_{B_2} < N_{B_1}$.

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SINGLE PHOTON ENCODINGS

Dual-rail qubit

- Photons have many degrees of freedom (polarization, spatial, angular momentum etc.).
- Each dof can described by set of mode operators $\{\hat{a}_k\}_{k=1}^{M}$ where M is the number of orthogonal modes
- Generally focus on two modes $k \in \{1, 2\}$ to define a photonic qubit. Logical states 0 and 1 are single-photon states

$$|0\rangle = \hat{a}_1^\dagger |\text{vac}\rangle \quad \text{and} \quad |1\rangle = \hat{a}_2^\dagger |\text{vac}\rangle$$

s.t. general **dual-rail qubit** $\Psi \in \text{span}\{|0\rangle, |1\rangle\}$

- Technically, $|\text{vac}\rangle = |\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$, $\hat{a}_1^\dagger |\text{vac}\rangle = \hat{a}_1^\dagger \otimes \hat{\mathbb{I}} |\text{vac}\rangle_1 \otimes |\text{vac}\rangle_2$

Single qubit operations

- Single-qubit operations implemented with **passive operations**.
- Passive operations commute with total photon number $\hat{N} = \sum_{k=1}^{2} \hat{a}_k^\dagger \hat{a}_k$
- Consist of unitary beam splitters and phase-shifters, $\hat{U}_{\mathrm{BS}}$ and $\hat{U}_\phi$, with Hamiltonians

$$\hat{H}_{\mathrm{BS}} = i\theta \mathrm{e}^{i\varphi} \hat{a}_1^\dagger \hat{a}_2 + \mathrm{h.c.},$$

$$\hat{H}_\phi = \sum_{k=1}^{2} \phi_k \hat{a}_k^\dagger \hat{a}_k.$$

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Exercise 2: Passive operations

Q: Show that any Hamiltonian of the form $\hat{H} = \sum_{i,j} H_{ij} \hat{a}_i^\dagger \hat{a}_j$ commutes with the total photon number operator $\hat{N} = \sum_{k=1}^{2} \hat{a}_k^\dagger \hat{a}_k$.

A: Equivalent to showing $\sum_k \left[\hat{a}_k^\dagger \hat{a}_k, \hat{a}_i^\dagger \hat{a}_j\right] = 0$. Use $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\,\hat{B}$ and $[\hat{a}_i^\dagger, \hat{a}_j] = -\delta_{ij}$.

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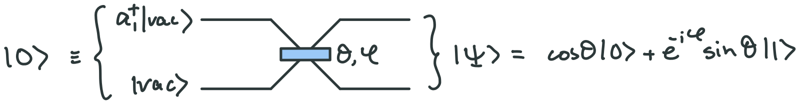

- Action of general **beamsplitter** on mode operators,

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix}}_{\equiv V_{\mathrm{BS}}} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

  where $V_{\mathrm{BS}}^{\dagger} V_{\mathrm{BS}} = \mathbb{I}$ and $\det V_{\mathrm{BS}} = 1$. Can create with Mach-Zehnder.
- Easy to show that

$$|0\rangle \xrightarrow{V_{\mathrm{BS}}} \cos\theta\,|0\rangle + \mathrm{e}^{-i\varphi}\sin\theta\,|1\rangle\,,$$
$$|1\rangle \xrightarrow{V_{\mathrm{BS}}} -\mathrm{e}^{i\varphi}\sin\theta\,|0\rangle + \cos\theta\,|1\rangle\,.$$

Poll 3: Pauli-X with a beam splitter

Consider two input modes $\hat{a}_1$ and $\hat{a}_2$ into a general beam splitter transformation with outputs $\hat{a}_1'$ and $\hat{a}_2'$ given as,

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}.$$

Up to a global phase, can we implement the Pauli-X matrix $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with this transformation?

(A) Yes

(B) No

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Answer 3: Pauli-X with a beam splitter

Consider two input modes $\hat{a}_1$ and $\hat{a}_2$ into a general beam splitter transformation with outputs $\hat{a}_1'$ and $\hat{a}_2'$ given as,

$$\begin{pmatrix} \hat{a}_1' \\ \hat{a}_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \mathrm{e}^{i\varphi}\sin\theta \\ -\mathrm{e}^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}.$$

Up to a global phase, can we implement the Pauli matrix $X = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with this transformation?

(A) Yes

(B) No

Choose, e.g., $\theta = \varphi = \pi/2$. Substitute into rotation matrix above to find $\left(\begin{smallmatrix} 0 & \mathrm{e}^{i\pi/2} \\ \mathrm{e}^{i\pi/2} & 0 \end{smallmatrix}\right) \propto X$. This is because $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$, and $-\mathrm{e}^{-i\pi/2} = \mathrm{e}^{i\pi/2}$.

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Exercise 3: Hadamard

Q: Transformation matrices for phase shifts and beamsplitter,

$$V_\phi = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \quad \text{and} \quad V_{\text{BS}} = \begin{pmatrix} \cos\theta & e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix}.$$

What combination of phase-shifters and beamsplitters produces the Hadamard matrix, $H = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$?

A: Choose $\phi_1 = 0$, $\phi_2 = \pi$ s.t. $V_\phi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and choose $\theta = \pi/4$ and $\varphi = \pi/2$ s.t. $V_{\text{BS}} = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Then $H = V_\phi V_{\text{BS}}$.

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Spatial, polarization, and time-bin encodings

- When choosing photonic dof for encoding, questions to consider:
  - Is the dof easy to manipulate?
  - Is the dof robust to relevant noise sources?
  - If necessary, can we scale-up for quantum information processing with many photons?
- Answers to these questions depend on context.
- Common encodings:

  (i) *Spatial*: Photon with fixed frequency ω, polarization etc., but may traverse two distinct paths $k = 1, 2$. Interaction by overlapping paths at, e.g., beamsplitters. Phase shifts via path lengths s.t. $\phi_k = \omega L_k / c$.

  (ii) *Polarization*: Photon with fiexed frequency, spatial path etc., but may be in a superposition of polarization states. Horizontal H and vertical V polarization define logical states, $|0\rangle = |H\rangle$ and $|1\rangle = |V\rangle$. Birefringent materials implement single-photon operations.

  (iii) *Time-bin*: Photon with fixed frequency, polarization, spatial path etc., but may occupy two distinct time-binned intervals $k = e, l$ (e for early, l for late). Fast optical switches and delays implement single-photon operations.

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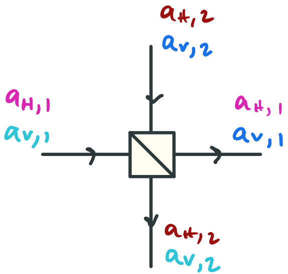
Swapping encodings: Polarization to spatial

- Swapping between encodings is possible
- E.g., given two polarization modes H, V and two spatial modes $1, 2$, implement a polarizing beamsplitter (PBS) s.t.

$$\hat{a}_{H,1} \to \hat{a}_{H,1} \quad \text{and} \quad \hat{a}_{H,2} \to \hat{a}_{H,2},$$
$$\hat{a}_{V,1} \to \hat{a}_{V,2} \quad \text{and} \quad \hat{a}_{V,2} \to \hat{a}_{V,1}.$$

- H gets transmitted while V gets reflected. Follow up by a polarization rotation results in swap from polarization qubit to spatial qubit



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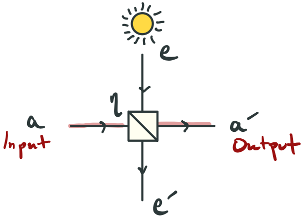
SINGLE PHOTON EVOLUTION

Single-photon evolution through thermal loss channel

- All communication links (i.e., quantum channels) are over noisy fibers or free-space, which can be modeled by thermal loss channels
- Focus: Thermal loss channel $\mathcal{L}_{\eta,N_E}$ on a single-photon state ρ_1
- Environment quanta N_E is the population of the environment (e.g., sun or background lights for free-space links), whereas loss probability $1-\eta$ is equal to absorption probability of the medium.
- E.g., given a fiber of length L, $\eta = \mathrm{e}^{-\alpha L}$ where α is an attenuation coefficient (typically quoted in dB/km). The exponential attenuation is a consequence of the Beer-Lambert law for absorptive media.



Thermal loss: Channel decomposition

- Consider a thermal-loss channel $\mathcal{L}_{\eta,N_E}$ which has the following decomposition $\mathcal{L}_{\eta,N_E} = \mathcal{A}_{G,0} \circ \mathcal{L}_{\tau,0}$ with

$$\tau G = \eta \quad \text{and} \quad \frac{G-1}{1-G\tau} = N_E.$$

- Parameters τ and G are related to η and N_E via

$$G = (1-\eta)N_E + 1 \quad \text{and} \quad \tau = \frac{\eta}{(1-\eta)N_E + 1}.$$

- To show decomposition:

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Thermal loss: Operator-sum representation

- Consider Kraus operators $\{\hat{A}_\ell\}_{\ell=0}^{\infty}$ of pure-loss channel $\mathcal{L}_{\tau,0}$

$$\hat{A}_\ell = \sqrt{\frac{(1-\tau)^\ell}{\ell!}}\tau^{\hat{a}^\dagger\hat{a}/2}\hat{a}^\ell.$$

- Consider Kraus operators $\{\hat{B}_k\}_{k=0}^{\infty}$ of quantum-limited amplifier $\mathcal{A}_{G,0}$,

$$\hat{B}_k = \sqrt{\frac{1}{k!}\frac{1}{G}\left(\frac{G-1}{G}\right)^k}\hat{a}^{\dagger k}G^{-\hat{a}^\dagger\hat{a}/2}.$$

- Using $\mathcal{L}_{\eta,N_E} = \mathcal{A}_{G,0}\circ\mathcal{L}_{\tau,0}$, thermal loss channel then has

$$\mathcal{L}_{\eta,N_E}(\rho) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\hat{B}_k\hat{A}_\ell\rho\hat{A}_\ell^\dagger\hat{B}_k^\dagger.$$

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Thermal loss: single-photon input

$\mathcal{L}_{\eta,N_E}(\rho_1) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\hat{B}_k\hat{A}_\ell\rho_1\hat{A}_\ell^\dagger\hat{B}_k^\dagger$

- Consider single-photon input ρ_1 with output $\mathcal{L}_{\eta,N_E}(\rho_1)$.
- Terms $\hat{A}_\ell\rho_1\hat{A}_\ell^\dagger$ are only non-zero when $\ell = 0, 1$. Thus,

$$\hat{A}_0\rho_1\hat{A}_0^\dagger = \tau\rho_1 \quad \text{and} \quad \hat{A}_1\rho_1\hat{A}_1^\dagger = (1-\tau)\,|\mathrm{vac}\rangle\langle\mathrm{vac}|\,.$$

- With probability τ, the photon is transmitted. With probability $1-\tau$, the photon is lost.

Poll 4: Pure loss and erasure cont.

When acting on a single-photon state ρ_1, the pure-loss channel $\mathcal{L}_\tau$ is equivalent to an erasure channel $\mathcal{L}_\varepsilon$ with erasure probability $\varepsilon = 1 - \tau$. What is the erasure state in this case? [Hint: Note that we are *losing* photons via loss.]

(A) Vacuum state

(B) Completely mixed single-photon state

(C) State with ≥ 2 photons

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Answer 4: Pure loss and erasure cont.

When acting on a single-photon state ρ_1, the pure-loss channel $\mathcal{L}_\eta$ is equivalent to an erasure channel $\mathcal{L}_\varepsilon$ with erasure probability $\varepsilon = 1 - \eta$. What is the erasure state in this case? [Hint: Note that we are *losing* photons via loss.]

(A) Vacuum state

(B) Completely mixed single-photon state

(C) State with ≥ 2 photons

Explicitly, $\mathcal{L}_\tau(\rho_1) = \tau\rho_1 + (1-\tau)\,|\mathrm{vac}\rangle\langle\mathrm{vac}|$. With probability τ, the photon is transmitted, and with probability $1-\tau$, the photon is lost.

Thermal loss: single-photon input cont.

$\mathcal{L}_{\eta,N_E}(\rho_1) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \hat{B}_k\hat{A}_\ell\rho_1\hat{A}_\ell^\dagger\hat{B}_k^\dagger$

- More complicated for amplifier $\mathcal{A}_{G,0}$ due to adding photons
- Relevent operators are $\hat{B}_k$ for $k = 0, 1$,

$$\hat{B}_0 = \sqrt{\frac{1}{G}G^{-\hat{a}^\dagger\hat{a}/2}} \quad \text{and} \quad \hat{B}_1 = \sqrt{\frac{1}{G}\left(\frac{G-1}{G}\right)}\hat{a}^\dagger G^{-\hat{a}^\dagger\hat{a}/2}.$$

- Appending to pure-loss channel leads to,
  1. $\hat{B}_0\hat{A}_0\rho_1\hat{A}_0^\dagger\hat{B}_0^\dagger = \frac{\tau}{G^2}\rho_1$; photon is unaffected by the channel
  2. $\hat{B}_1\hat{A}_0\rho_1\hat{A}_0^\dagger\hat{B}_1^\dagger = \frac{2(G-1)}{G^3}\tau\rho_2$; one noisy photon added to state.
  3. $\hat{B}_0\hat{A}_1\rho_1\hat{A}_1^\dagger\hat{B}_0^\dagger = \frac{(1-\tau)}{G}\,|\text{vac}\rangle\langle\text{vac}|$; photon is just lost.
  4. $\hat{B}_1\hat{A}_1\rho_1\hat{A}_1^\dagger\hat{B}_1^\dagger = \frac{G-1}{G^2}(1-\tau)\rho_1$; photon is lost and replaced with a single noisy photon state ρ_1.

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Thermal loss: single-photon input cont.

$\mathcal{L}_{\eta,N_E}(\rho_1) = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty} \hat{B}_k \hat{A}_\ell \rho_1 \hat{A}_\ell^\dagger \hat{B}_k^\dagger$

- Overall

$$\mathcal{L}_{\eta,N_E}(\rho_1) = \left[\frac{\tau}{G^2} + \frac{G-1}{G^2}(1-\tau)\right]\rho_1 + \frac{(1-\tau)}{G}\,|\mathrm{vac}\rangle\langle\mathrm{vac}| + \frac{(G-1)^2 + 2\tau(G-1)}{G^2}\rho_{\geq 2},$$

where $\rho_{\geq 2}$ is a quantum state with more than two photons.

- Transmission event probabilities,

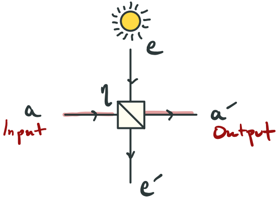
$$p_{\text{success}} = \frac{\tau}{G^2} = \frac{\eta}{[(1-\eta)N_E + 1]^3} \quad \text{(successful transmission)}$$

$$p_{\text{random}} = \frac{G-1}{G^2}(1-\tau) = \frac{(1-\eta)^2 N_E(N_E+1)}{[(1-\eta)N_E+1]^3} \quad \text{(random photon)}$$

$$p_{\text{vac}} = \frac{(1-\tau)}{G} = \frac{(1-\eta)(N_E+1)}{[(1-\eta)N_E+1]^2} \quad \text{(receive nothing)}$$

$$p_{\geq 2} = 1 - p_{\text{success}} - p_{\text{depolarizing}} - p_{\text{vac}} \quad \text{(receive} \geq 2 \text{ photons)}$$

Exercise 4: Low thermal noise



Q: Assume $N_E \ll 1$. Expand p_{success}, p_{random}, and p_{vac} to first order in N_E. Show that $p_{\geq 2} = 2\eta(1-\eta)N_E + \mathcal{O}\left(N_E^2\right)$. Intuitively explain result?

A: Expanding previous expressions,

$$p_{\text{success}} = \frac{\eta}{\left[(1-\eta)N_E + 1\right]^3} \approx \eta\left(1 - 3N_E(1-\eta)\right) + \mathcal{O}(N_E)^2$$

$$p_{\text{random}} = \frac{(1-\eta)^2 N_E (N_E + 1)}{\left[(1-\eta)N_E + 1\right]^3} \approx (1-\eta)^2 N_E + \mathcal{O}\left(N_E^2\right)$$

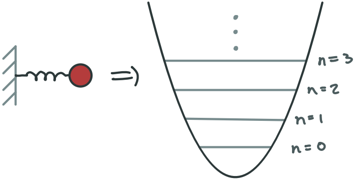
$$p_{\text{vac}} = \frac{(1-\eta)(N_E + 1)}{\left[(1-\eta)N_E + 1\right]^2} \approx (1-\eta)\left(1 - (1-2\eta)N_E\right) + \mathcal{O}\left(N_E^2\right).$$

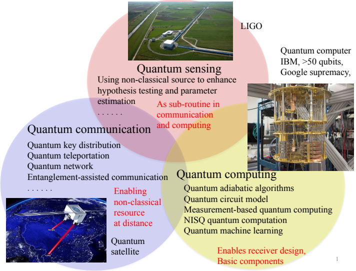
en, $p_{\geq 2} = 1 - p_{\text{success}} - p_{\text{random}} - p_{\text{vac}} \approx 2\eta(1-\eta)N_E + \mathcal{O}\left(N_E^2\right)$

RECAP AND EXIT SURVEY

Center for Quantum Networks NSF-ERC

- Discussed how photons are good for just about anything (sensing, computing, communication)
- Reviewed general description of evolution and quantum channels (Kraus operators, purification, unitary extension)
- Analyzed Gaussian bosonic channels (loss, amplifier, AGN) and their influence at single-photon level
- Surveyed single-photon encodings and single-qubit operations (e.g., passive operations on dual-rail qubit)





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Course Evaluation Survey

We value your feedback on all aspects of this short course. Please go to the link provided in the Zoom Chat or in the email you will soon receive to give your opinions of what worked and what could be improved.

CQN Winter School on Quantum Networks

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