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Quantum Networks
NSF Engineering Research Center

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Information in a Photon

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— University of Arizona

CQN Winter School on Quantum Networks
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Quantum information processing

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- Communications
  - Deep-space lasercom
  - Quantum security: QKD, covert communications
- Computing
  - Factoring [breaks RSA, Diffie-Hellman]
  - Search and optimization
  - Blind quantum computing
  - Multiparty privacy-preserving computations
  - Simulations of complex quantum systems
- Sensing
  - Passive imaging (astronomy, microscopy, SDA)
  - Active photonic sensors (RF photonic antennas, fiber-optic gyroscopes, beam deflectometers, surface topography, endoscopy, lidars, …)
  - Gravitational wave sensors (LIGO – squeezed light)
  - Magnetic field sensing (Neuronal imaging, chip testing)

Photonic quantum information processing

- Wherever light gets used in encoding, extracting, carrying or processing information [communications, sensing, imaging, computing, simulations, …], what is the best performance permissible by the laws of quantum physics?

Quantum optics meets Information theory

Quantum optics – quantum theory of light and its interaction with matter

- Non-linear optics
- Atomic systems
- Many-body physics

Information theory – quantifying “information” in the context of communications, sensing and computing

- Detection theory
- Estimation theory
- Data compression
- Channel coding

Course objective

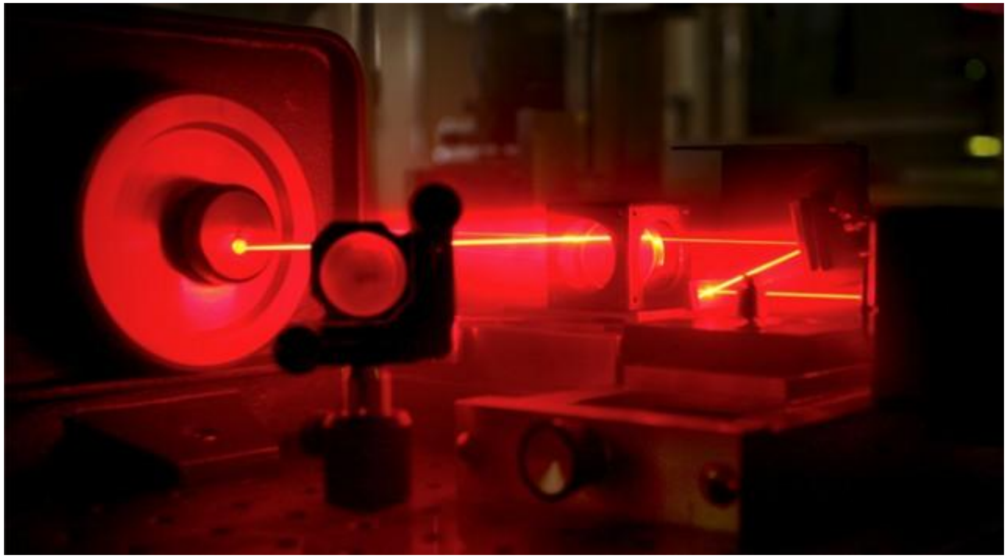
- Introduction to some of the mathematical tools necessary to understand the quantum representation of classical laser light, and certain *non-classical* states of light, including single-photon and multi-photon states and *squeezed* states of light
- Using these tools to understand how to manipulate (classical and quantum) light using *interference*
- Using basic tools from quantum information theory to uncover applications of quantum techniques (sources of light, ways to manipulate light, and detection schemes) for improved performance in encoding, transmission and processing of information

Course plan

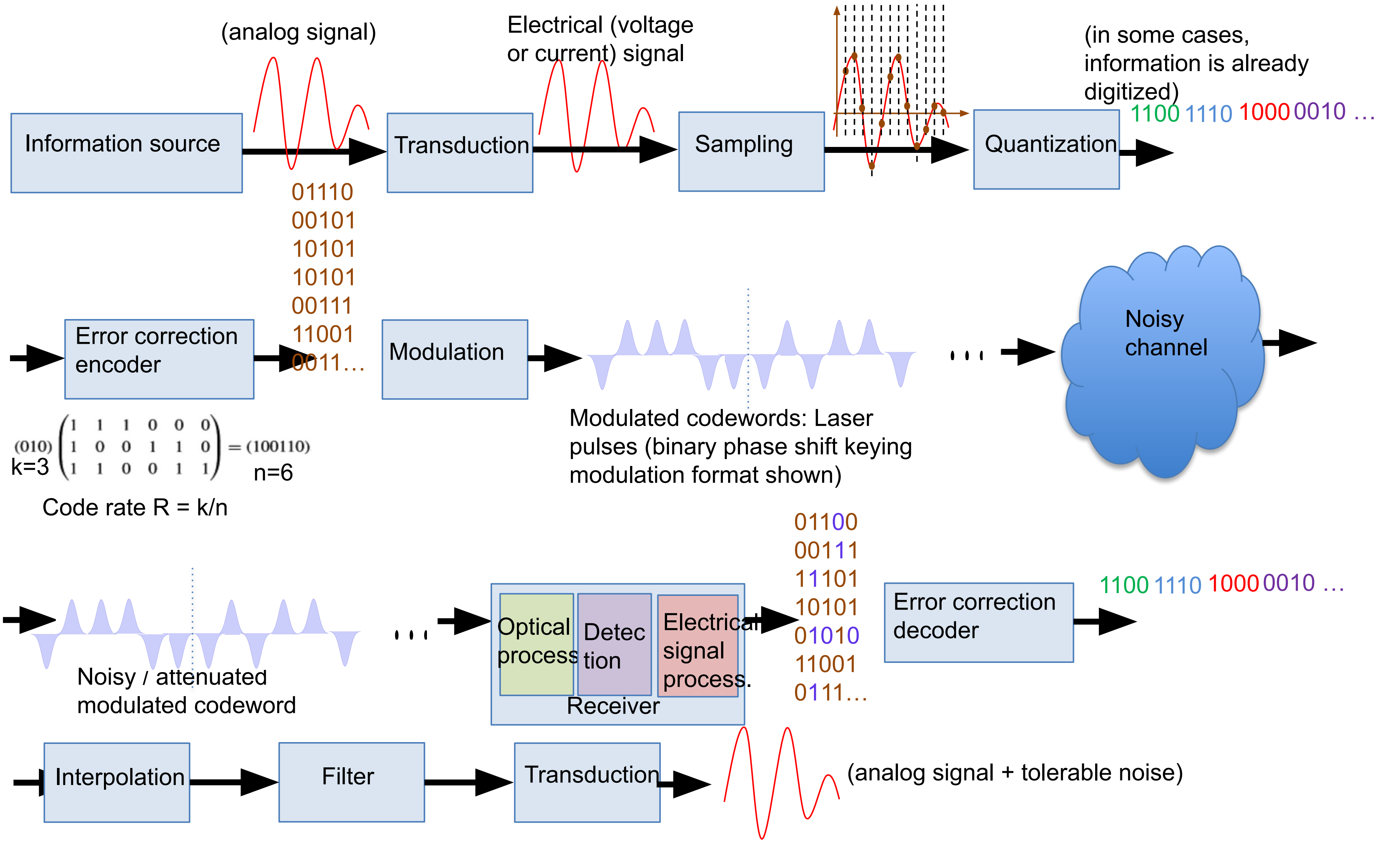
- **Module 1:** Quantum limits of information encoded in laser light [Saikat Guha]
- **Module 2:** Quantum information advantage arising from interfering photons [Christos Gagatsos]

- Multiple choice questions interspersed through the course, to be answered through zoom polls
- Post-lecture survey

Module 1: Quantum limits of information encoded in laser light



Digital (classical) communications



“Quantum limit” of an optical receiver

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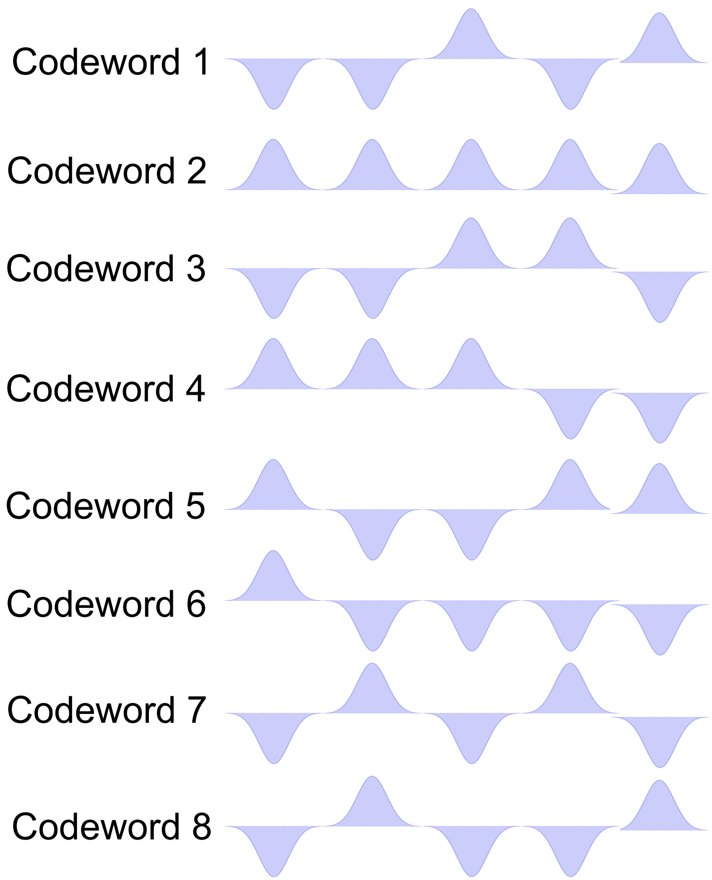

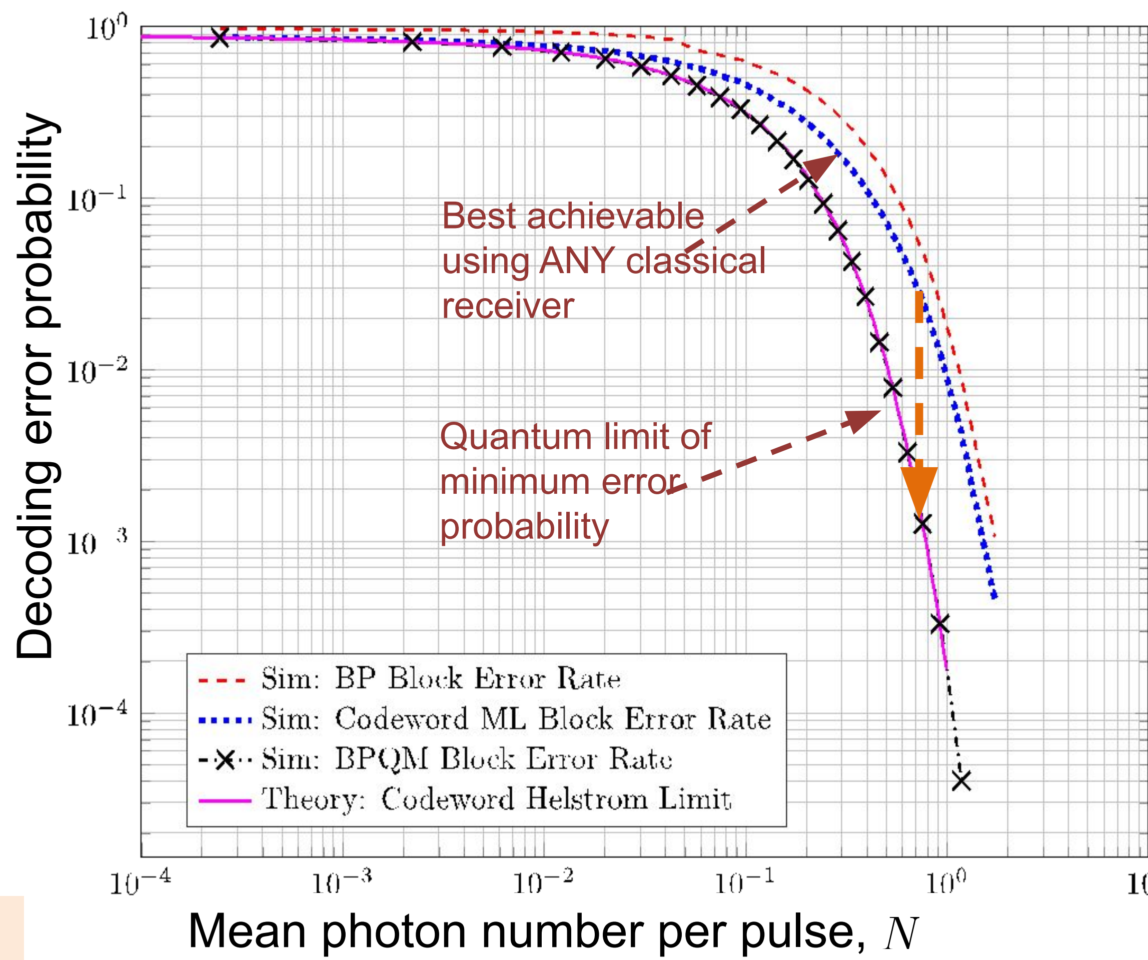

Optical receiver is a “mini quantum computer”

Silva, SG, Dutton, Phys. Rev. A 87, 052320 (2013)
Rengaswamy, Seshadreesan, SG, Pfister, Nature npj Quantum Inf 7, 97 (2021)
Delaney, Seshadreesan, MacCormack, Galda, SG, Narang, Phys. Rev. A 106, 032613 (2022)

Random process describing *Discrete* events in *continuous* time and/or space

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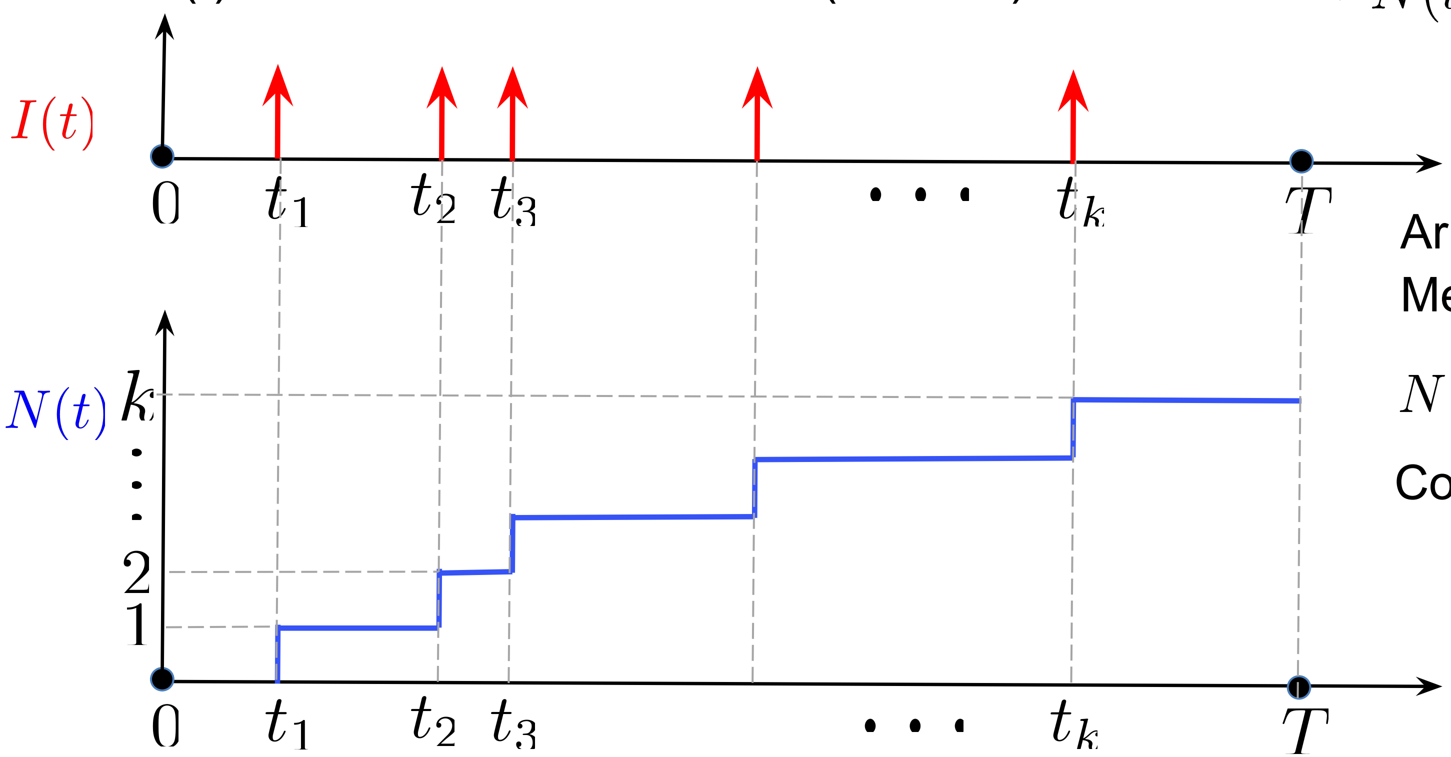
- Where such random processes might occur
  - Spiking patterns in neurons, bank teller queue, photon detection, …
- Arrival process, $I(t)$; counting process, $N(t)$
  - I(t): random arrivals (each arrival denoted by a delta function)
  - N(t): number of occurrences (arrivals) before time *t*, $N(t) = \int_0^t I(t)dt$



Arrival rate: $\lambda(t)$

Mean # arrivals:

$$N = \int_0^T \lambda(t)dt$$

Constant rate:

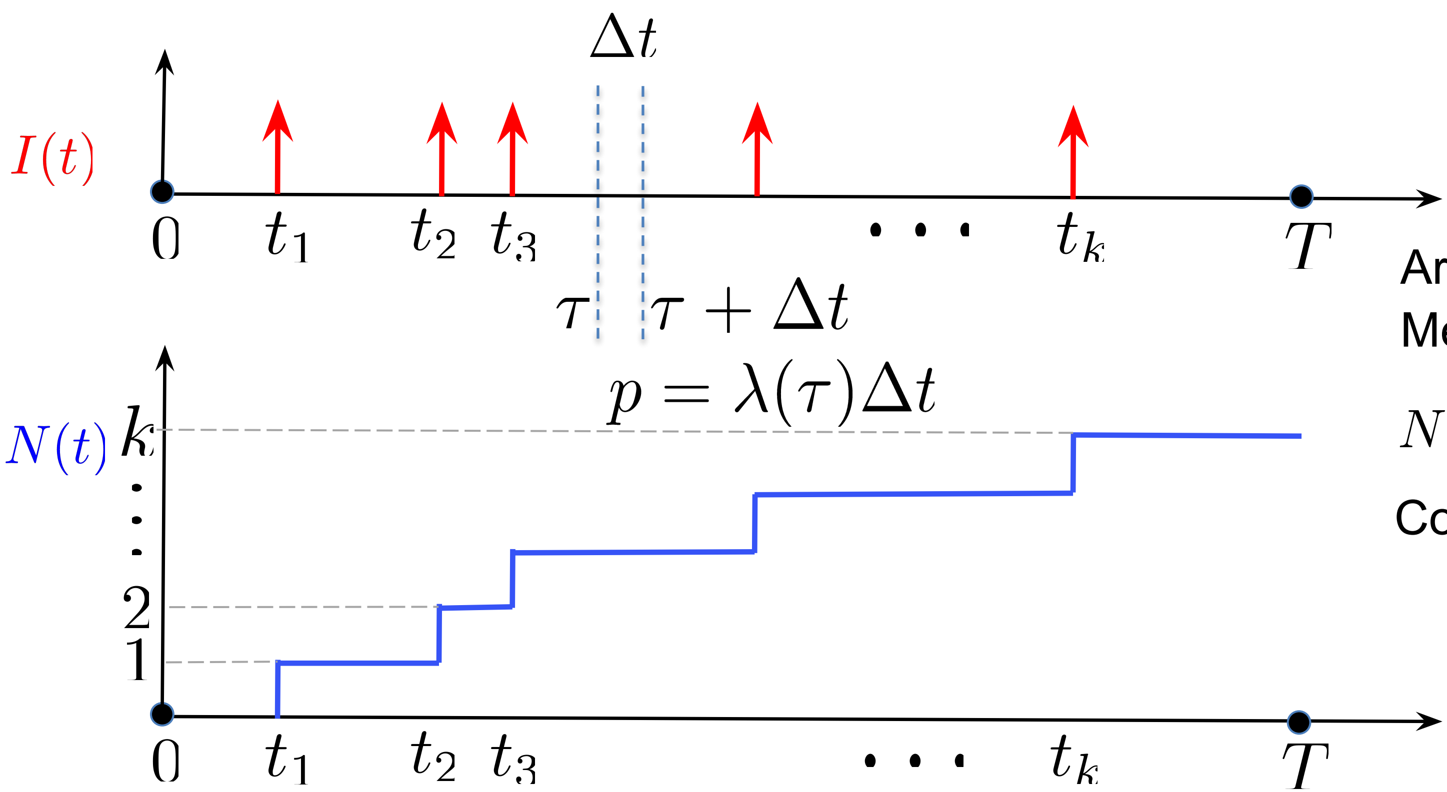
$$\lambda(t) = \lambda$$

$$N = \lambda T$$

Poisson point process (PPP)

1. Probability of an arrival in a tiny time step is equal to the rate of the arrival process (at that time) times the size of the time increment

2. Probabilities of arrivals in disjoint time steps are statistically independent



Arrival rate: $\lambda(t)$

Mean # arrivals:

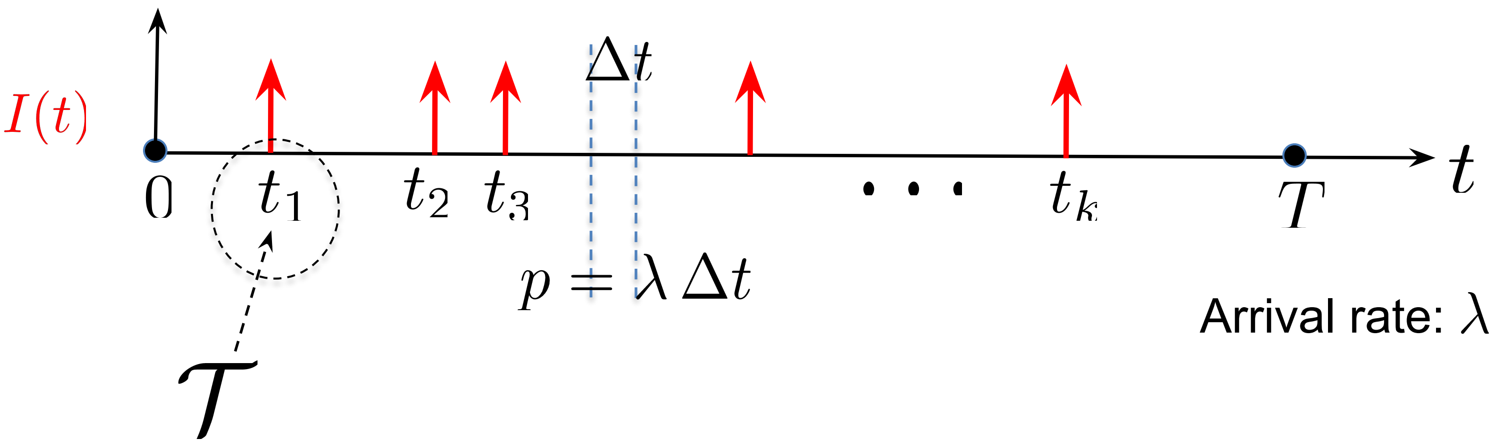
$$N = \int_0^T \lambda(t)dt$$

Constant rate:

$$\lambda(t) = \lambda$$

$$N = \lambda T$$

Probability distribution of inter-arrival time



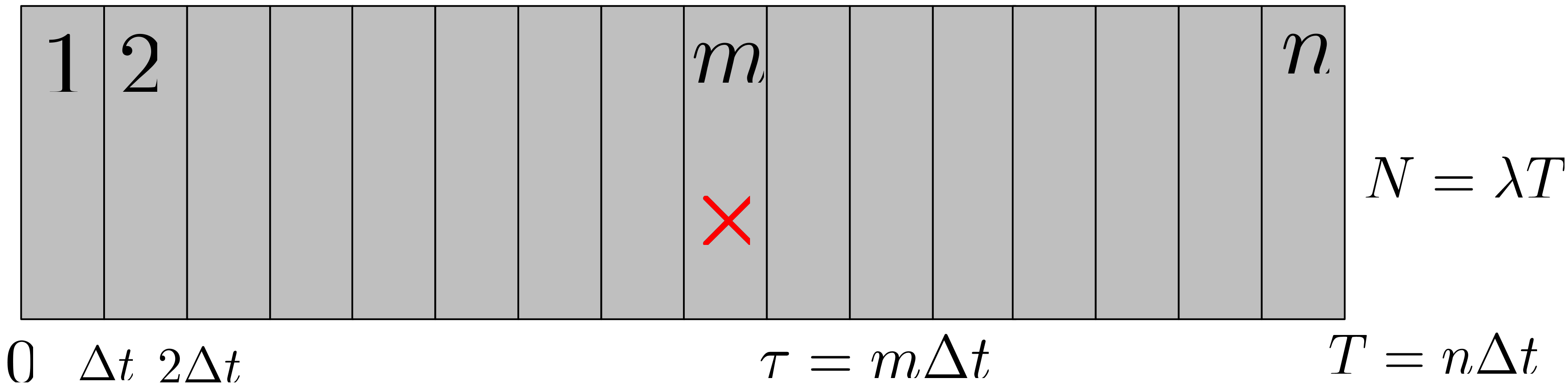
Time of first arrival

Arrival rate: $\lambda(t) = \lambda$

Probability distribution of the time of first arrival, $P_{\mathcal{T}}(\tau),\ \tau \geq 0$

Probability distribution of the total number of arrivals, $P_K[k],\ k = 0, 1, \ldots$

Inter-arrival times and number of arrivals



Probability of one arrival in a Δt interval, $p = \lambda \Delta t$

Let us denote by t, the time of first arrival; $P[t = \tau] = (1-p)^{m-1}p; \ \ m = 1, 2, \ldots$

c.d.f., $F_{\mathcal{T}}(\tau) = P[t \leq \tau] = \sum_{j=1}^{m} (1-p)^{j-1}p = 1 - (1-p)^m$

$$= 1 - (1 - \lambda\Delta t)^{\tau/\Delta t} \xrightarrow[\Delta t \to 0]{} 1 - e^{-\lambda\tau}$$

$$m \to \infty \qquad \frac{T}{\tau} = \frac{n}{m}$$
$$n \to \infty$$

p.d.f., $P_{\mathcal{T}}(\tau) = \frac{d}{d\tau} F_{\mathcal{T}}(\tau) = \lambda e^{-\lambda\tau}; \ \tau \geq 0$

Probability distribution of the total number of arrivals K, $P_K[k] = \binom{n}{k} p^k (1-p)^{n-k}$

$$\xrightarrow[\Delta t \to 0]{} ?$$

Distribution of the number of arrivals

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$$\tau = m\Delta t \qquad T = n\Delta t$$

$$p = \lambda\,\Delta t \qquad N = \lambda T$$

Probability distribution of the total number of arrivals K, $P_K[k] = \binom{n}{k} p^k (1-p)^{n-k}$

$$\xrightarrow[\Delta t \to 0]{} ?$$

Problem 1: What is the distribution of the total number of arrivals K? (k=0, 1, 2, …)

A: $P_K[k] = (1 - 1/N)^k (1/N)$
B: $P_K[k] = e^{-N} N^k / k!$
C: $P_K[k] = N^{k^2} - N^{(k+1)^2}$
D: I do not know.

A pulse of laser light

- Quasi-mono-chromatic laser light pulse: in √ (photons/m[2]-sec) units

$$\tilde{E}(\boldsymbol{r},t) = E(\boldsymbol{r},t)e^{-j\omega_0 t+\phi}, t \in (0,T], \boldsymbol{r} \in \mathcal{A}$$

$$= \psi(\boldsymbol{r})s(t)e^{-j\omega_0 t+\phi}, \boldsymbol{r} \equiv (x,y)$$

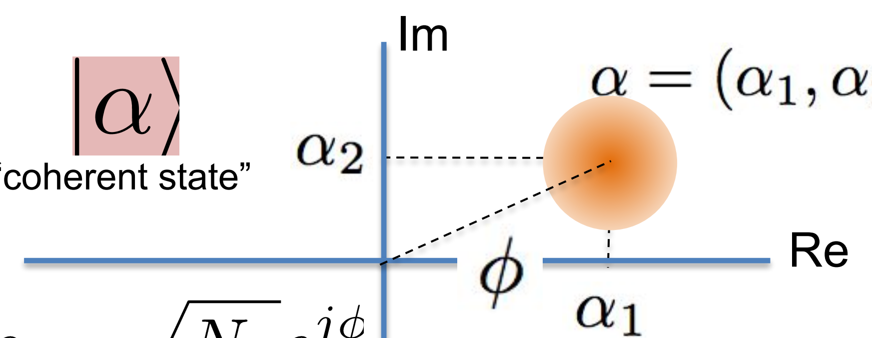
Spatial and temporal dependence may not be factorable in general

Spatial mode ($\psi(\boldsymbol{r})$) Temporal mode ($s(t)$)

- Mean photon number, $N_S = \int_0^T \int_{\mathcal{A}} |\tilde{E}(\boldsymbol{r},t)|^2 d\boldsymbol{r}dt$

$$= \int_0^T \int_{\mathcal{A}} |E(\boldsymbol{r},t)|^2 d\boldsymbol{r}dt$$

$|\alpha\rangle$ "coherent state"



$$\alpha = \sqrt{N_S}e^{j\phi}$$

$$N_S = |\alpha|^2$$

Phase space picture: once we identify a spatio-temporal-polarization mode, a complex number describes the state of the laser pulse

No detector can accurately measure the field $E(\boldsymbol{r},t)$

Mode (space / time / polarization)

- An optical mode is the "shape" of a confined EM field in space, time and polarization (the three independent degrees of freedom of the photon)
- Time & Frequency are the same degree of freedom (related by Fourier transform)

$$\phi_\nu(\boldsymbol{r}, t);\ \boldsymbol{r} \in \mathcal{A}, t \in [0, T), \nu = 1, 2$$

$$\int_{\mathcal{A}} \int_0^T \phi_\nu(\boldsymbol{r}, t) \phi_\nu^*(\boldsymbol{r}, t) d\boldsymbol{r} dt$$

$$= \int_{\mathcal{A}} \int_0^T |\phi_\nu(\boldsymbol{r}, t)|^2 \, d\boldsymbol{r} dt = 1$$

We will take a mode to be normalized

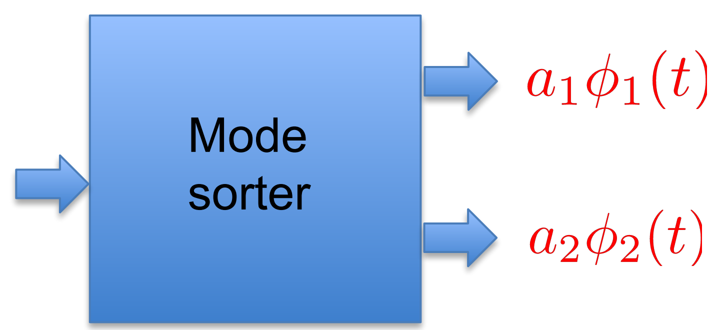
Orthogonal modes

- Two modes $\phi_\nu(\boldsymbol{r}, t)$ and $\psi_\mu(\boldsymbol{r}, t)$ are **orthogonal** if,

$$\int_{\mathcal{A}}\int_0^T \phi_\nu(\boldsymbol{r}, t)\psi_\mu^*(\boldsymbol{r}, t)d\boldsymbol{r}dt = 0$$

  - If $\nu \neq \mu$, the two modes will be orthogonal regardless of their spatial and temporal shapes
  - When $\nu = \mu$, we will drop the polarization subscript
  - When we say two "temporal modes" $s_1(t)$ and $s_2(t)$ are orthogonal, we will implicitly assume the mode functions being referred to have the same spatial modes and polarization

$$E(t) = a_1\phi_1(t) + a_2\phi_2(t) + 0\phi_3(t) + \ldots$$



Maximum number of orthogonal modes

- Consider temporal modes, $\phi_k(t), k = 1, \ldots K$
- …and their Fourier transforms,

$$\Phi_k(f) = \int \phi_k(t) e^{-2\pi j f t} dt$$

$$\phi_k(t) = \int \Phi_k(f) e^{2\pi j f t} df$$

- How many (K) orthogonal modes $\phi_k(t)$ can be "fit into" a time-bandwidth product of T x W? i.e.,
  - $\phi_k(t) = 0, t \notin [0, T)$, and $\Phi_k(f) = 0,\ f - f_0 > \left|\frac{W}{2}\right|$
  - While ensuring orthogonality: $\int_0^T \phi_k(t)\phi_l^*(t) dt = \delta_{kl}$
- Answer: $K \approx WT$, and these optimal mode functions are "Prolate Spheroidal" functions
- All of above holds for spatial modes as well

Slepian, D. Prolate spheroidal wave functions, Fourier analysis and uncertainty — IV: Extension to many dimensions; Generalized prolate spheroidal functions Bell Syst. Tech. J., vol. 43, pp. 3009-3057, Nov. 1964.

Some intuition: choices of WT modes

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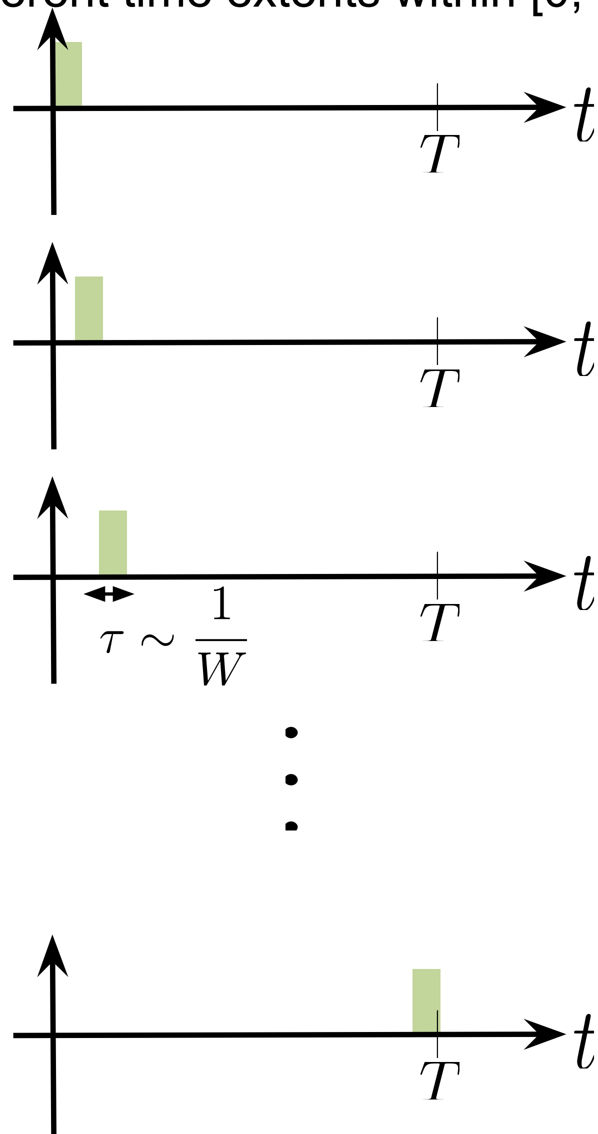
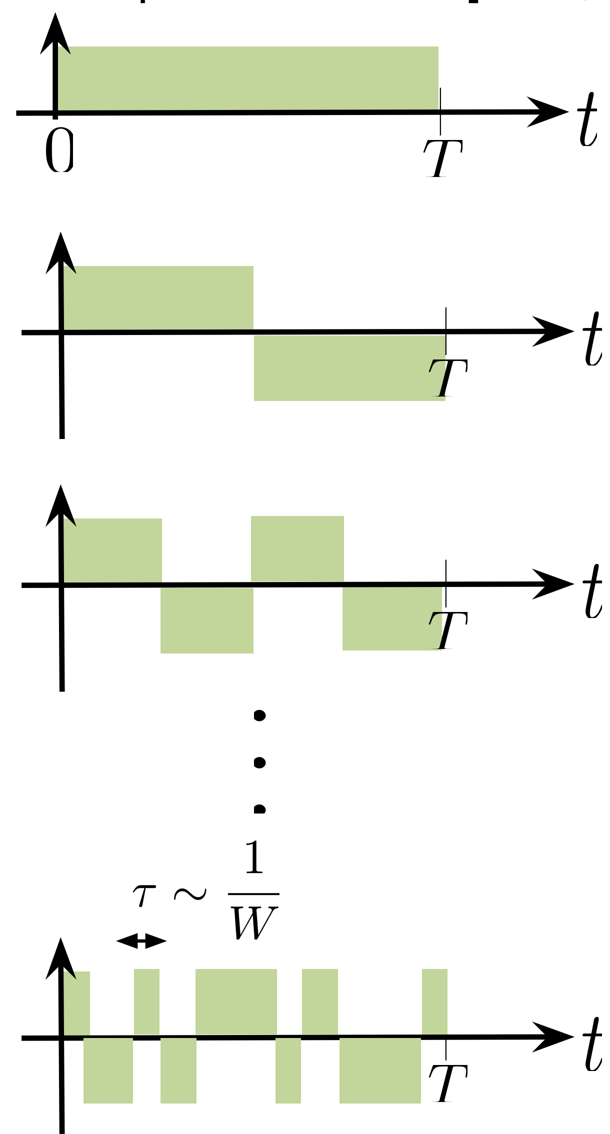
Each mode fills up the same BW, but different time extents within [0, T)



Each mode fills up the same time [0, T), but different freq extents within [-W/2, W/2]



Example: flat-top temporal mode

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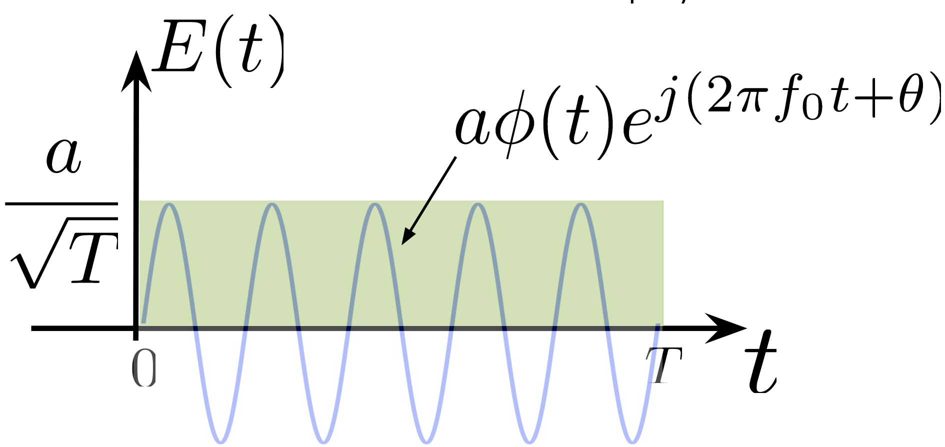
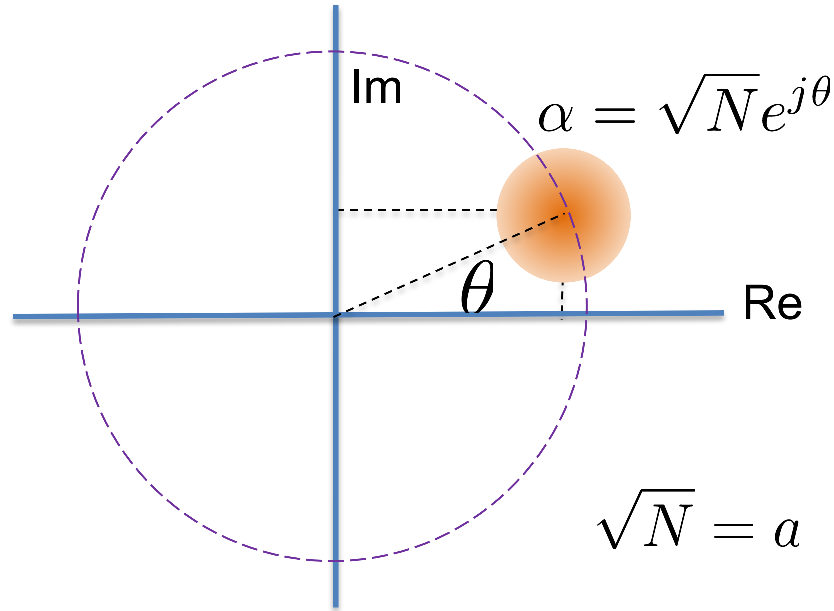
$$\phi(t) = \begin{cases} \frac{1}{\sqrt{T}}, & t \in [0, T) \\ 0, & \text{otherwise} \end{cases}$$

Coherent state of this mode: $|\alpha\rangle$

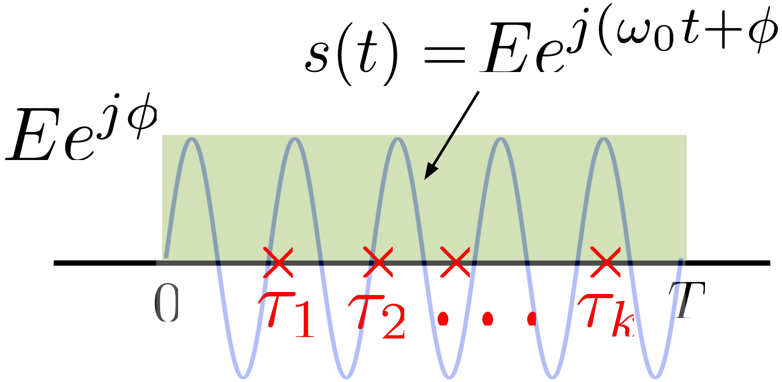


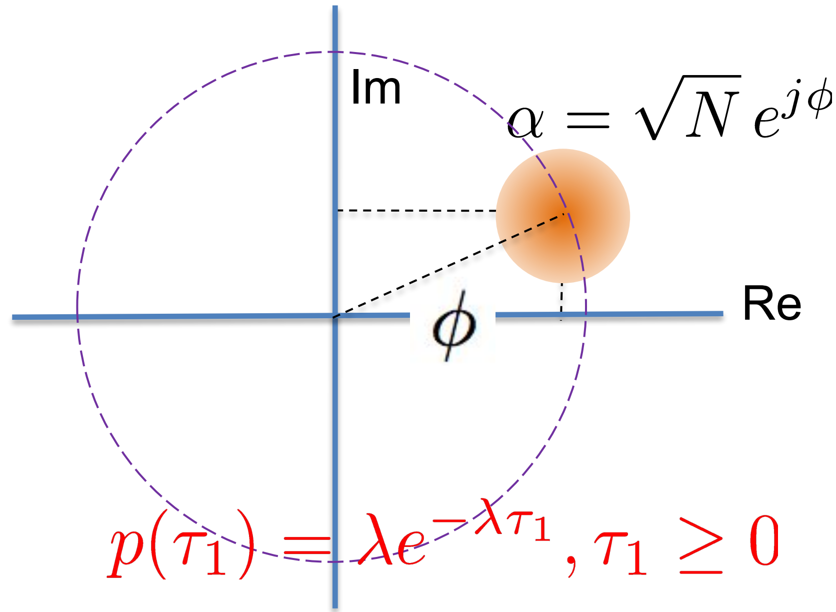
Phase-space representation:



Photon detection on a coherent state of the flat-top mode (a “square pulse”)

- PPP of constant rate



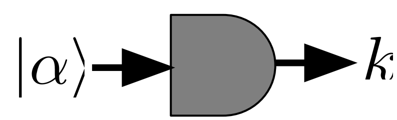


$$p(\tau_1) = \lambda e^{-\lambda \tau_1}, \tau_1 \geq 0$$

$$s(t) = \begin{cases} Ee^{j\phi}, & t \in [0, T], \\ 0, & \text{otherwise} \end{cases}$$

$$p_K[k] = \frac{e^{-N} N^k}{k!}, k \in \mathbb{Z}$$

$$\lambda(t) = |s(t)|^2 = \lambda = E^2, t \in (0, T]$$



Mean photon number in pulse, $N = \int_0^T \lambda(t)dt = \lambda T = E^2 T$

On-off keying (OOK) modulation

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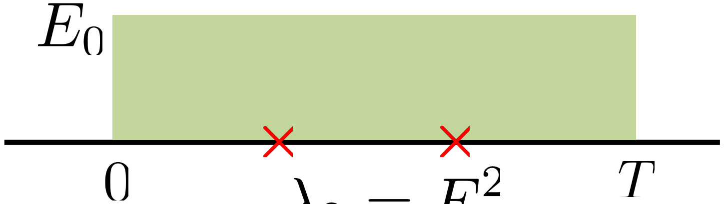
H_1 $|0\rangle$

$$\lambda_1 = 0$$
$$N_1 = 0$$

$|\alpha\rangle$ H_2



$$\lambda_2 = E_0^2$$
$$N_2 = E_0^2 T = |\alpha|^2 = N$$

assume equal priors:

$$P(\mathrm{H}_1) = P(\mathrm{H}_2) = \frac{1}{2}$$

"Maximum Likelihood" decision rule:

k = 0 click □ say H_1

k > 0 clicks □ say H_2

Waiting for the first click suffices

$$P_e = P(\mathrm{H}_1)P(\mathrm{H}_2|\mathrm{H}_1) + P(\mathrm{H}_2)P(\mathrm{H}_1|\mathrm{H}_2)$$
$$= \frac{1}{2}P(k > 0|\mathrm{H}_1) + \frac{1}{2} \times P(k = 0|\mathrm{H}_2)$$

Probability of > 0 photon arrivals if H_1 is actually true (N = 0 photon pulse incident on detector)

Probability of 0 photon arrivals if H_2 is actually true (N photon pulse incident on detector)

$$= \frac{1}{2} \times 0 + \frac{1}{2} \times e^{-N}$$
$$= \frac{1}{2}e^{-N}$$

Recall the Poisson distribution, evaluate at k=0:

$$P_K[k] = \frac{e^{-N}N^k}{k!}, k \in \mathbb{Z}$$

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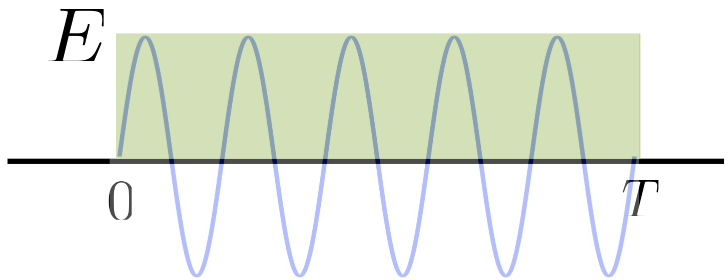
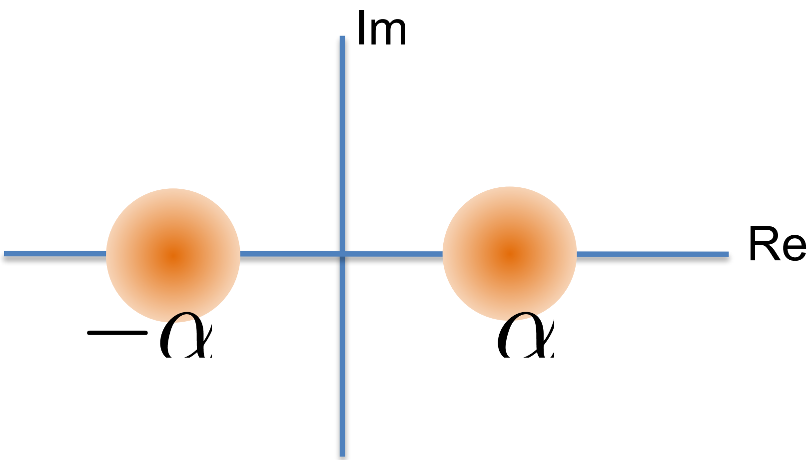
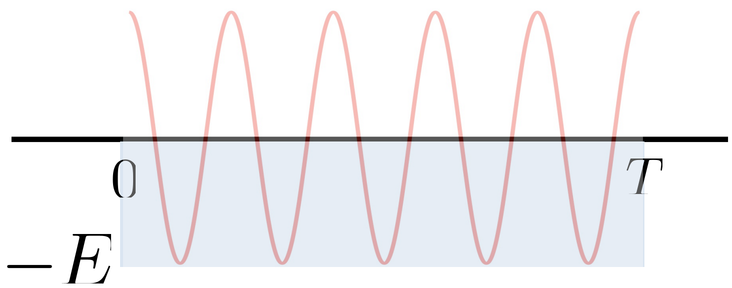
Binary phase shift keying (BPSK)

$Ee^{j\phi}$ with, $\phi = \{0, \pi\}$





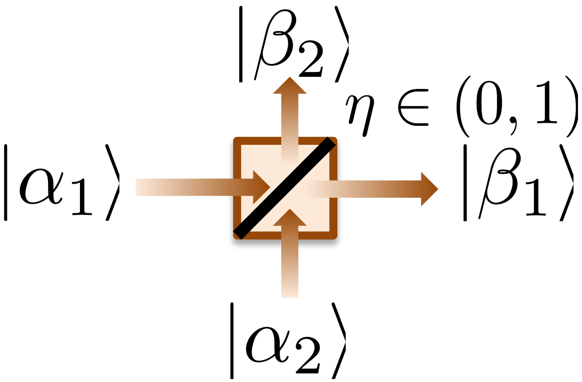


Mean photon number in the pulse is the same for either state:

$$N = |\alpha|^2 = E^2 T$$

Interference (passive linear optics)

- Beamsplitter



$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = U \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Transmissivity, $\eta = \cos^2\theta$
Phase, $\phi \in (0, 2\pi]$

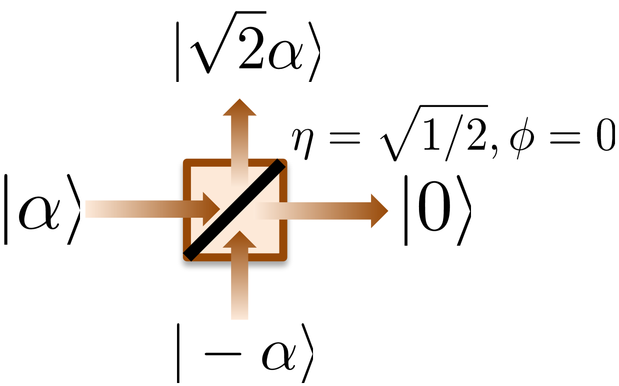
$$U(\theta, \phi) = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ \sin\theta & -e^{i\phi}\cos\theta \end{pmatrix}$$

$$\beta_1 = \sqrt{\eta}\alpha_1 + e^{i\phi}\sqrt{1-\eta}\alpha_2$$
$$\beta_2 = \sqrt{1-\eta}\alpha_1 - e^{i\phi}\sqrt{\eta}\alpha_2$$

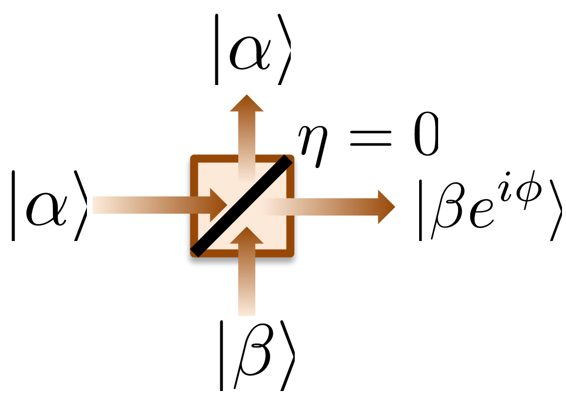
$$\boldsymbol{\beta} = U\boldsymbol{\alpha}$$

- Examples

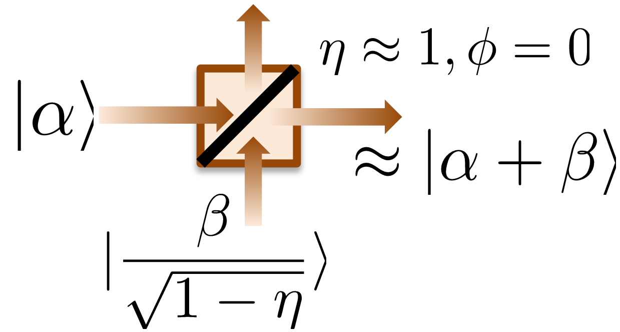
Destructive interference



Pure phase



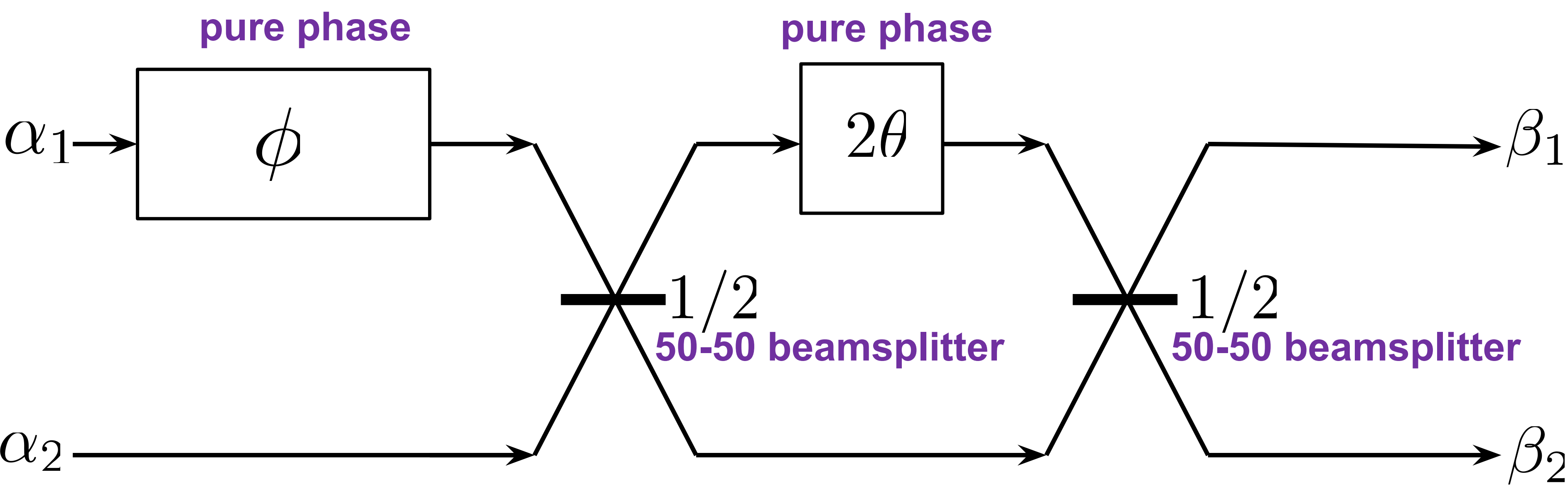
Nulling (displacement)

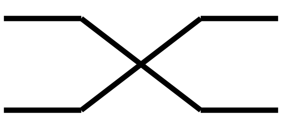


Mach-Zehnder interferometer (MZI)

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = U \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$U(\theta, \phi) = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ \sin\theta & -e^{i\phi}\cos\theta \end{pmatrix}$$

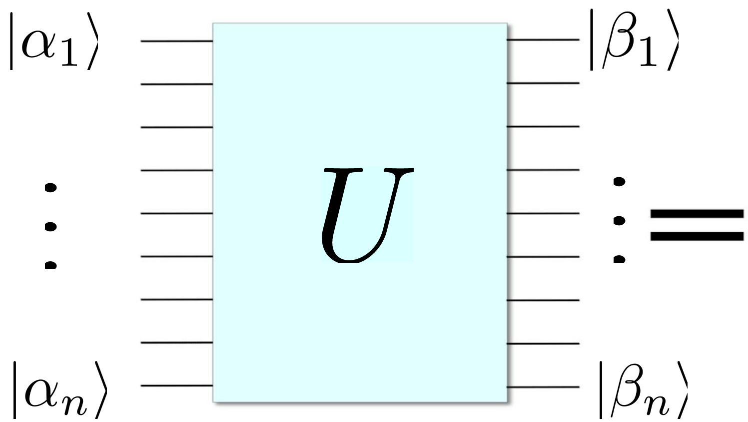




$U(\theta, \phi)$

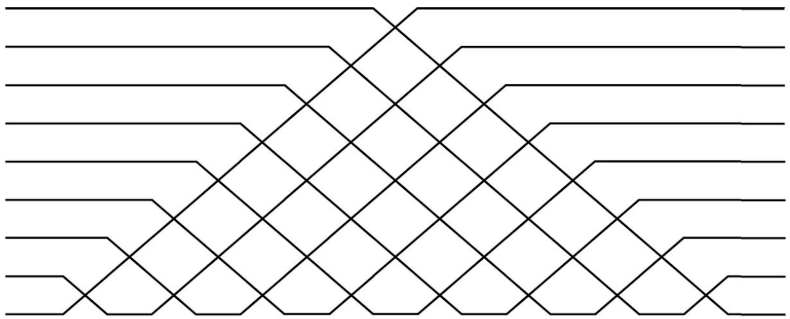
Arbitrary N-mode linear optical unitary

- Any N-by-N unitary U can be realized with M = N(N-1)/2 MZIs. Therefore, one needs tuning N(N-1) phases to realize any U



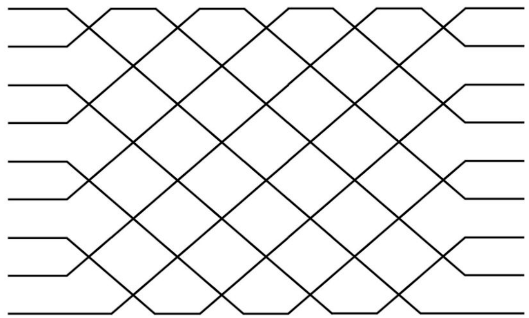
$$\boldsymbol{\beta} = U\boldsymbol{\alpha}$$

a

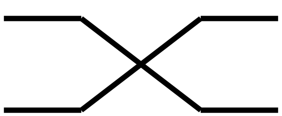


Reck et al., PRL 73, 1 (1994)

b

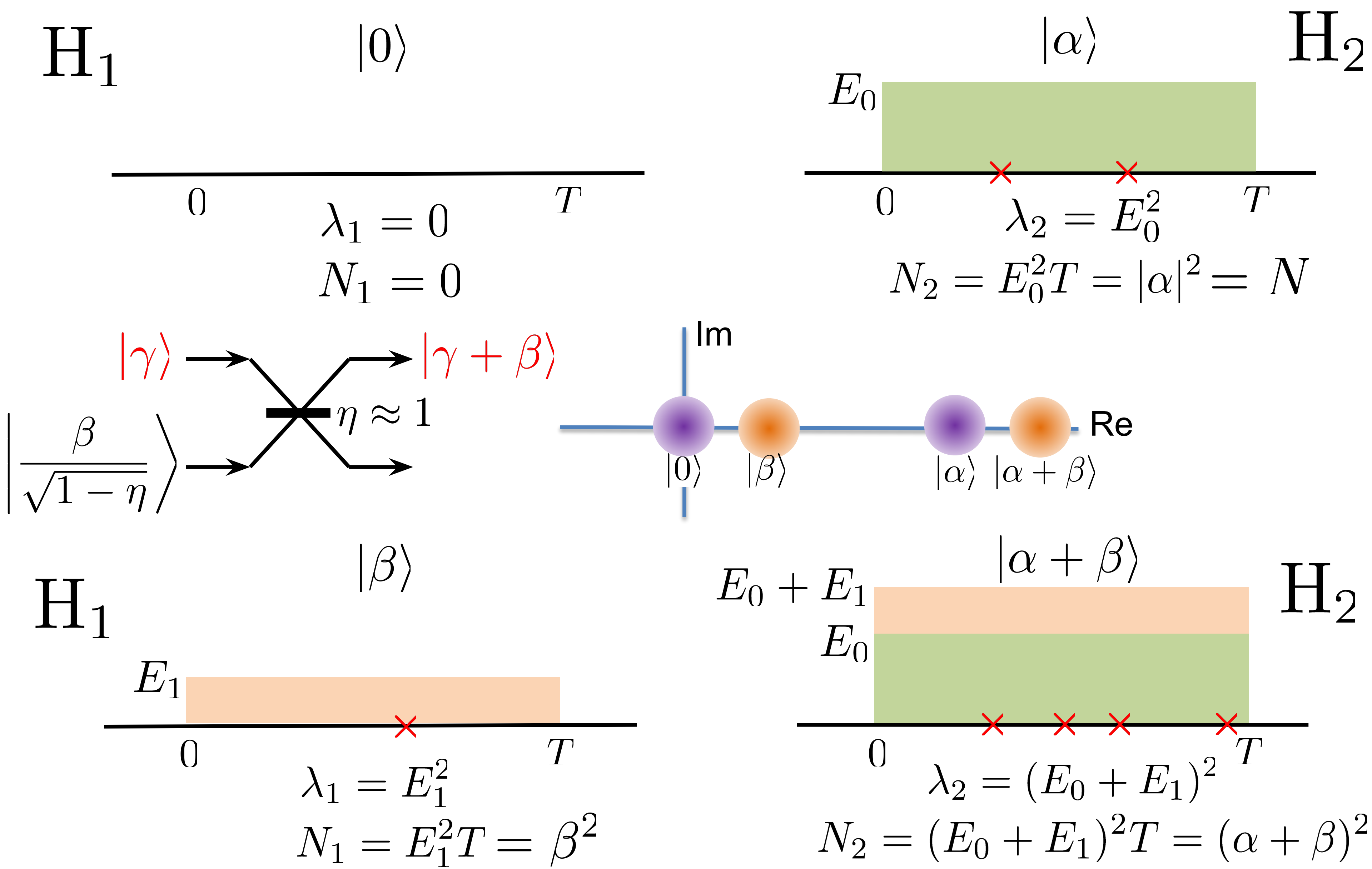


Clements et al., Optica 3 (12), 1460-1465 (2016)



$$U(\theta, \phi)$$

On-off keying (OOK), Kennedy receiver

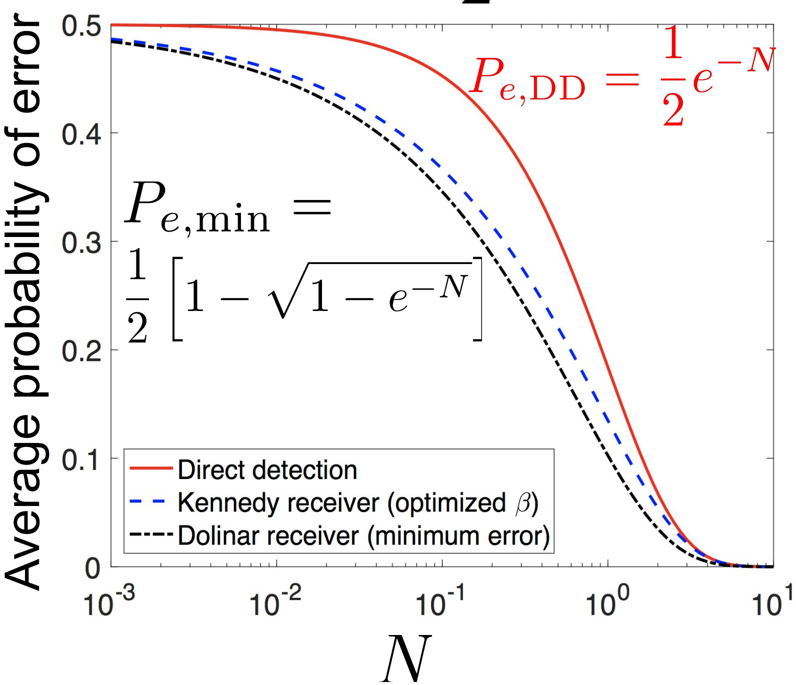


P(error) for the Kennedy receiver

- Let us use the same decision rule
  - click = "on", no-click = "off"

$$P_e = P(\mathrm{H}_1)P(\mathrm{H}_2|\mathrm{H}_1) + P(\mathrm{H}_2)P(\mathrm{H}_1|\mathrm{H}_2)$$
$$= \frac{1}{2}P(k>0|\mathrm{H}_1) + \frac{1}{2} \times P(k=0|\mathrm{H}_2)$$
$$= \frac{1}{2}\left(1 - e^{-\beta^2}\right) + \frac{1}{2}e^{-(\alpha+\beta)^2}$$

Optimize (minimize) this over the choice of β



Discriminating BPSK pulses $\{|-\alpha\rangle, |\alpha\rangle\}$

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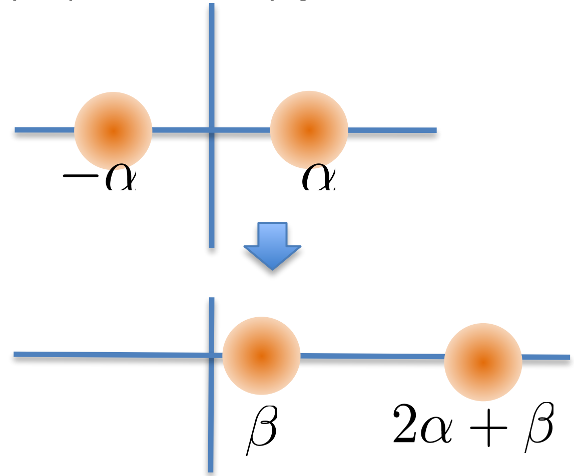
- Kennedy (requires perfect amplitude and phase reference)
  - Exact nulling $\{|-\alpha\rangle, |\alpha\rangle\} \longrightarrow \{|0\rangle, |2\alpha\rangle\}$, $|\alpha|^2 = N$

    R. Kennedy, MIT Research Laboratory for Electronics, Quarterly Progress Report 110, 219 – 225 (1972)
  - Optimized nulling $\{|-\alpha\rangle, |\alpha\rangle\} \longrightarrow \{|\beta\rangle, |2\alpha+\beta\rangle\}$

    Takeoka and Sasaki, Phys. Rev. A 78, 022320 (2008)

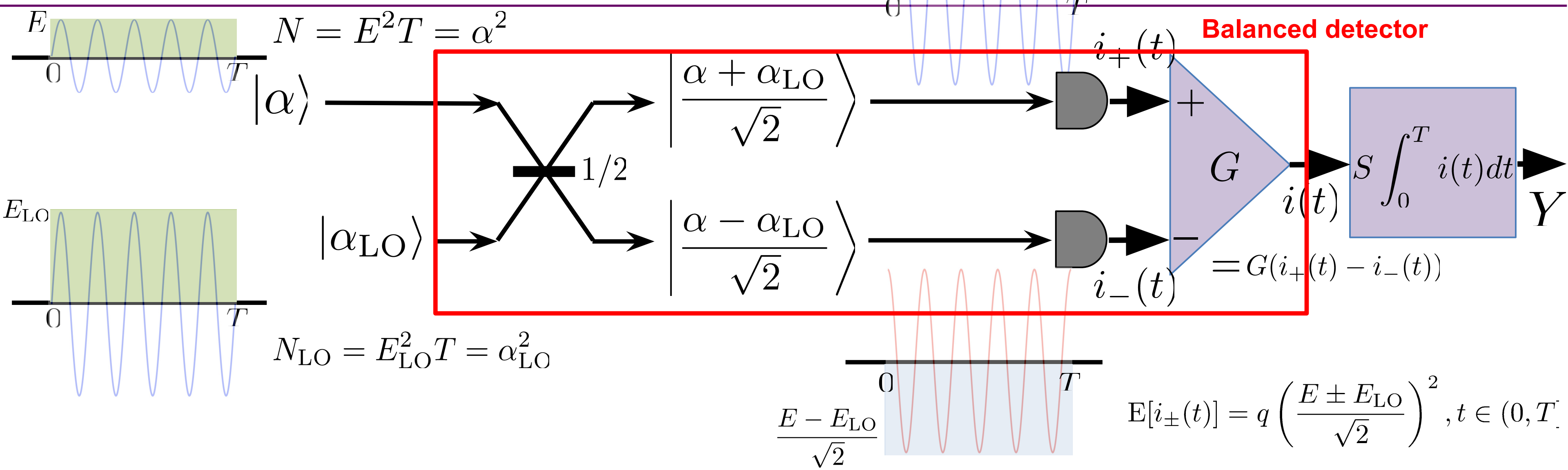


$$P_e(N) = \min_{\beta}\left[\frac{1}{2}e^{-(2\alpha+\beta)^2} + \frac{1}{2}\left(1 - e^{-\beta^2}\right)\right]$$

$$= \frac{1}{2}e^{-4N}, \ \beta = 0$$ (exact nulling, suboptimal)

Can we build a receiver for BPSK that does NOT require an amplitude reference?

Homodyne detection



Assume for now that both $\alpha, \alpha_{\mathrm{LO}}$ are real, and $N_{\mathrm{LO}} \gg N$

$$Y = qGS(K_+ - K_-)$$

$$K_+ \sim \mathrm{Poisson}(N_+) \sim \mathcal{N}(N_+, N_+);\ N_+ = \left|\frac{\alpha+\alpha_{\mathrm{LO}}}{\sqrt{2}}\right|^2$$

$$K_- \sim \mathrm{Poisson}(N_-) \sim \mathcal{N}(N_-, N_-);\ N_- = \left|\frac{\alpha-\alpha_{\mathrm{LO}}}{\sqrt{2}}\right|^2$$

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = qGS(N_+ - N_-) = 2qGS\alpha\alpha_{\mathrm{LO}}$$

$$\sigma^2 = (qGS)^2(N_+ + N_-) = (qGS)^2(\alpha^2 + \alpha_{\mathrm{LO}}^2)$$

$$\mathrm{Poisson}(\lambda) \approx \mathrm{Gaussian}(\lambda, \lambda),\ \lambda \gg 1$$

$$e^{-\lambda}\frac{\lambda^r}{r!} \simeq \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(r-\lambda)^2}{2\lambda}}$$

Homodyne detection: output distribution

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$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

$$\mu = qGS(N_+ - N_-) = 2qGS\alpha\alpha_{\mathrm{LO}}$$

$$\sigma^2 = (qGS)^2(N_+ + N_-) = (qGS)^2(\alpha^2 + \alpha_{\mathrm{LO}}^2)$$

Pick the scaling constant S such that the mean $\mu = \alpha$

Problem 2: What is the distribution of Y?

A: $Y \sim \mathcal{N}(\alpha, 1/4)$
B: $Y \sim \mathcal{N}(\alpha, 1/2)$
C: $Y \sim \mathcal{N}(\alpha, 1)$
D: I do not know.

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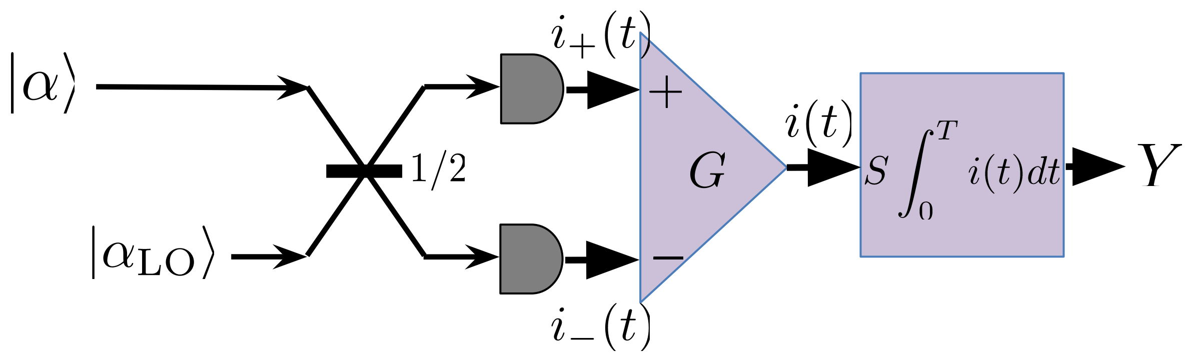
Break [5 minutes]

Homodyne detection (continued)

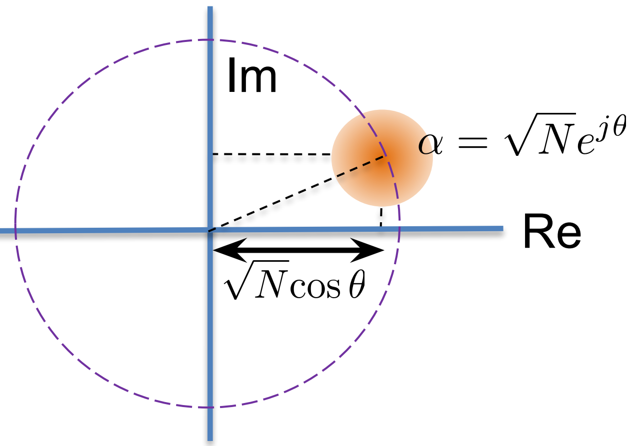
- Local Oscillator (LO) has a phase offset with the input pulse



$$\alpha = \sqrt{N}e^{j\theta}$$

$$\alpha_{\mathrm{LO}} = \sqrt{N_{\mathrm{LO}}}e^{j\theta_{\mathrm{LO}}}$$

$$N_{\mathrm{LO}} \gg N$$



Show that:

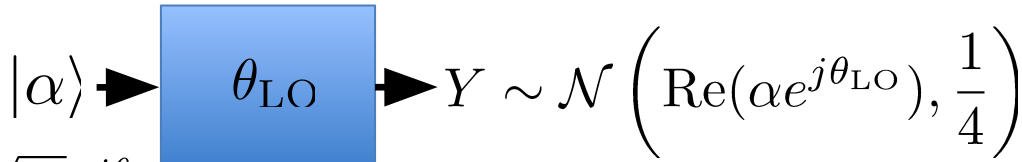
$$Y \sim \mathcal{N}\left(\mathrm{Re}(\alpha e^{j\theta_{\mathrm{LO}}}), \frac{1}{4}\right)$$

i.e.,

$$Y \sim \mathcal{N}\left(\sqrt{N}\cos(\theta + \theta_{\mathrm{LO}}), \frac{1}{4}\right)$$

Homodyne detection

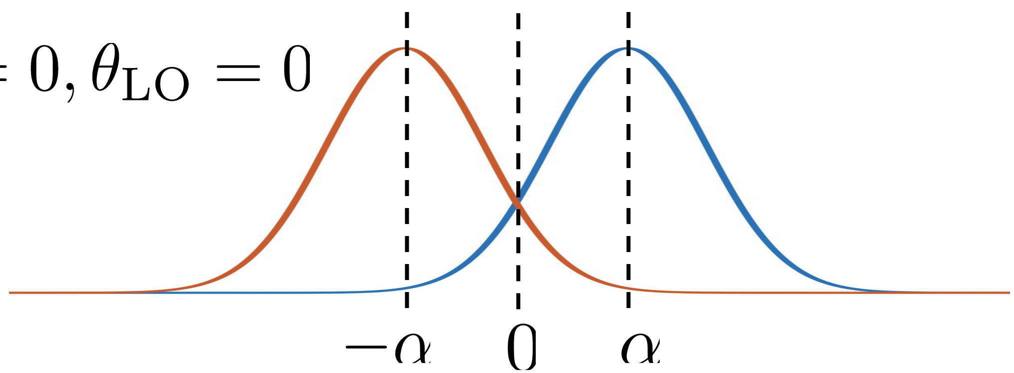
Black box description:



$$\alpha = \sqrt{N} e^{j\theta}$$

$$Y \sim \mathcal{N}\left(\mu, \sigma^2\right) \Rightarrow p_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-\mu)^2/2\sigma^2}$$ Gaussian probability distribution

$$\theta = 0, \theta_{\mathrm{LO}} = 0$$



$$P_e = \frac{1}{2}\mathrm{erfc}(\sqrt{2N})$$

Unambiguous BPSK state discrimination

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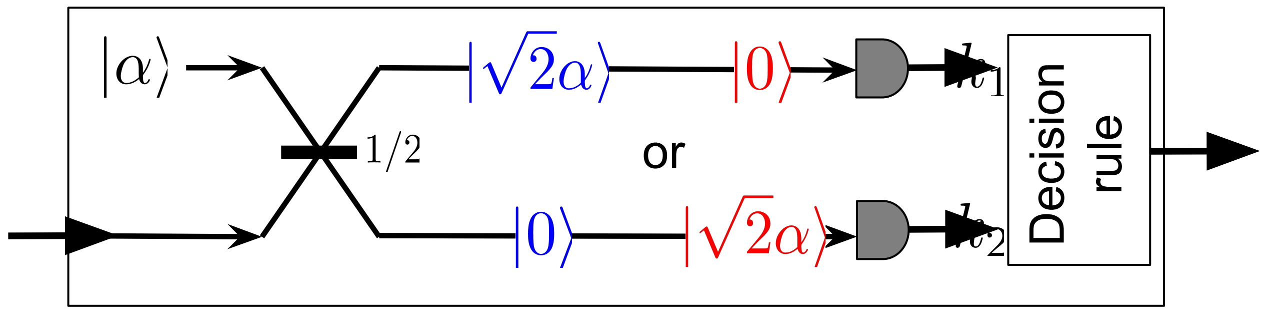
$|\alpha|^2 = N$

$X = 1$ $|\alpha\rangle$

$X = 2$ $|-\alpha\rangle$



$Y = 1$ $|\alpha\rangle$

$Y = 2$ $|-\alpha\rangle$

$Y = 3$ erasure

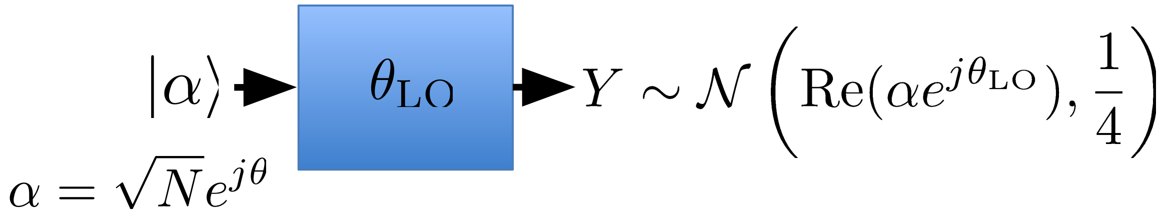
$$S := p_{Y|X}(y|x) =$$

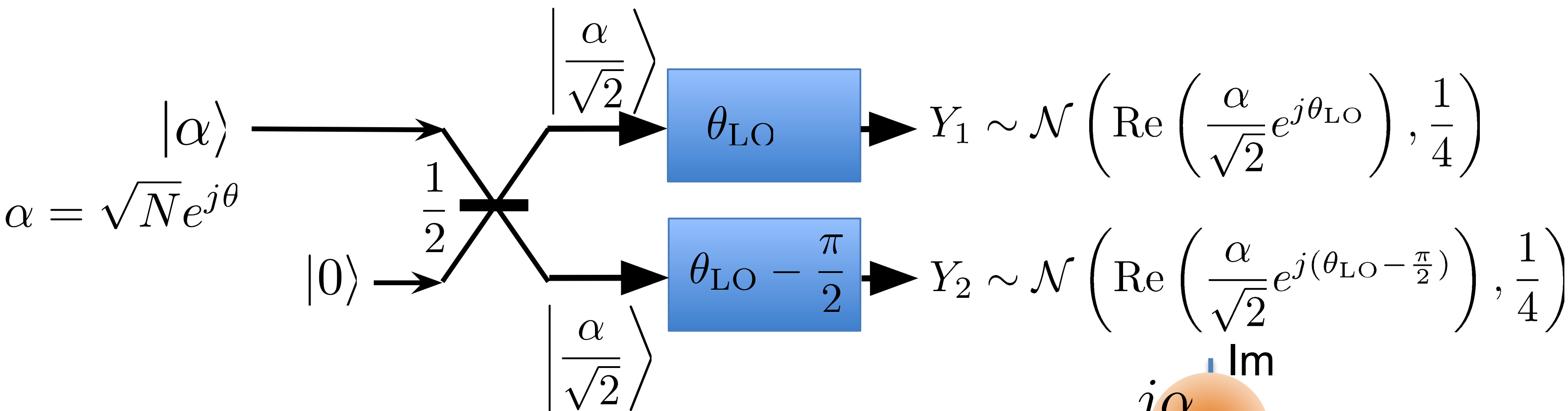
x \ y	1	2	3
1			
2			

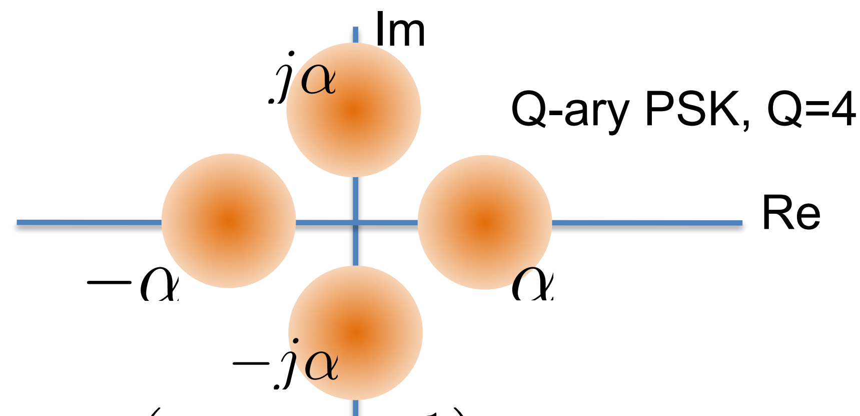
If forced to make a decision, i.e. by mapping the erasure to one possible input, then the average probability of error (assume equal priors), $P_{e,\mathrm{USD}} = e^{-2N}$

Dual homodyne detection (heterodyne)

Recall black box description:



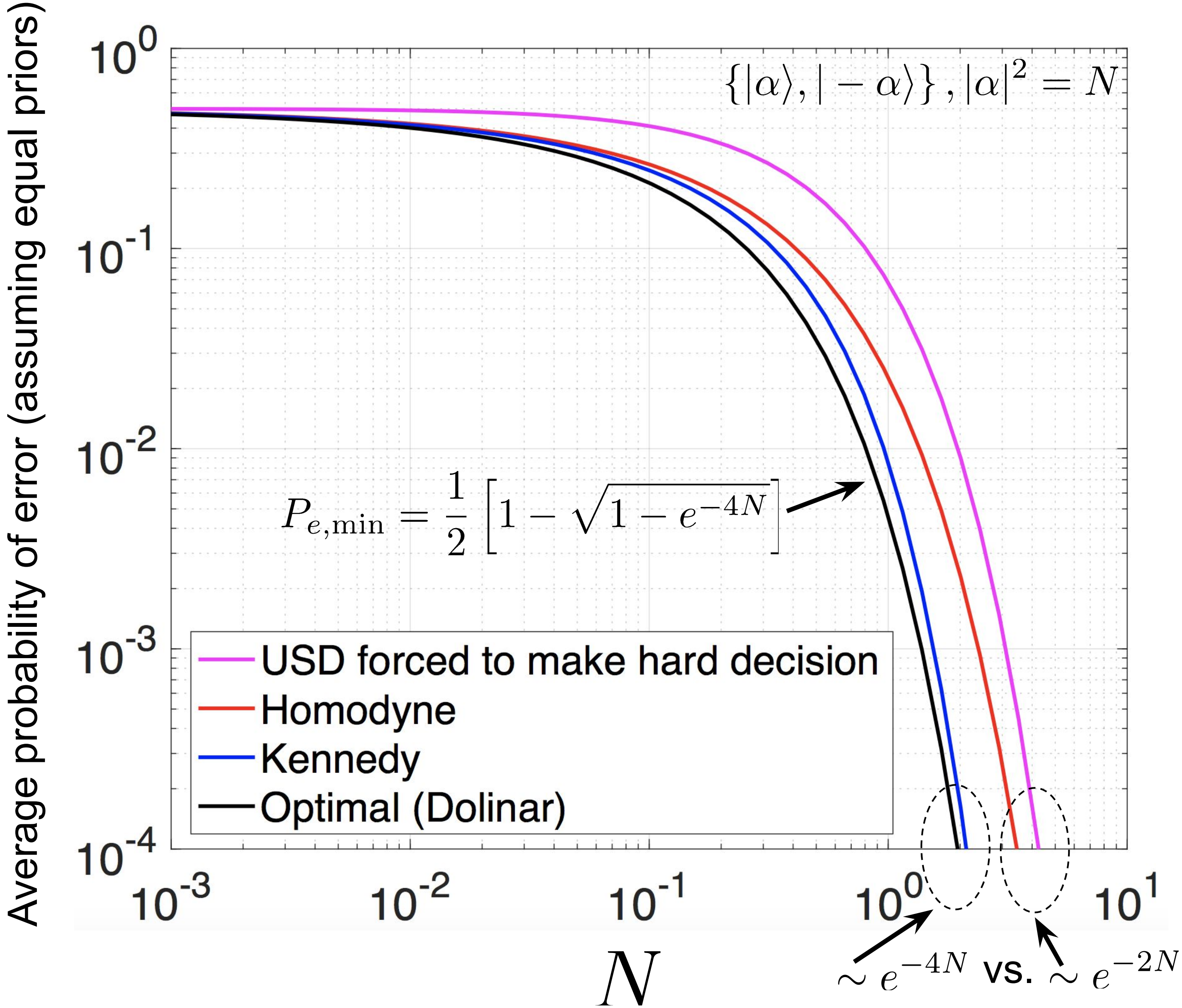




Define: $X_1 = Y_1\sqrt{2}, X_2 = Y_2\sqrt{2}$

We can show: $X_1 \sim \mathcal{N}\left(\mathrm{Re}(\alpha e^{j\theta_{\mathrm{LO}}}), \frac{1}{2}\right)$ $\quad X_2 \sim \mathcal{N}\left(\mathrm{Im}(\alpha e^{j\theta_{\mathrm{LO}}}), \frac{1}{2}\right)$

BPSK discrimination performance



Quantum states and measurements

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- A (pure) state is described by a unit-norm column vector $|\psi\rangle$
- Von Neumann (projective) measurement on a state is described by a set of unit-norm orthonormal vectors $\{|w_k\rangle\}$, $\langle w_k|w_j\rangle = \delta_{kj}$
- If the state $|\psi\rangle$ is measured, the k-th outcome appears with probability, $p_k = |\langle w_k|\psi\rangle|^2$
- If two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal, i.e., $\langle\psi_1|\psi_2\rangle = 0$ measurement $\{|w_1\rangle = |\psi_1\rangle, |w_2\rangle = |\psi_2\rangle\}$ achieves $P_e = 0$

Number (Fock) state of a mode

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Mode $\phi(t)$, a quantum system, is excited in a coherent state $|\alpha\rangle$, $\alpha \in \mathbb{C}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons K is a Poisson random variable of mean N

Mode $\phi(t)$, a quantum system, is excited in a number state $|n\rangle, n \in \{0, 1, \ldots, \infty\}$

If we do ideal direct detection of mode $\phi(t)$, the total number of photons K = n (exactly so; K is not a random variable).

A mode of ideal laser light is in a coherent state. Number (Fock) state of a given mode is VERY hard to produce experimentally

There are infinitely many other types of "states" of the mode $\phi(t)$. Coherent state and Fock state are just two example class of states

$|n\rangle, n \in \{0, 1, \ldots, \infty\}$ Fock states of a mode are special: they form an orthonormal basis that spans any general quantum state $|\psi\rangle$ of that mode

$$\langle m|n\rangle = \delta_{mn} \quad \text{and} \quad |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \ , \ \sum_{n=0}^{\infty} |c_n|^2 = 1$$

Coherent state as a quantum state

$$|\alpha\rangle = \sum_{n=0}^{\infty} \left(\frac{e^{-\frac{|\alpha|^2}{2}} \alpha^n}{\sqrt{n!}} \right) |n\rangle$$

$\leftarrow c_n$

$$\langle\alpha|\beta\rangle = \exp\left[\alpha^* \beta - \frac{1}{2}(|\alpha|^2 + |\beta|^2)\right] \neq 0$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \cdots$$

Ideal photon detection is a von Neumann quantum measurement described by projectors, $\{|n\rangle\langle n|\}$, $n = 0, 1, \ldots, \infty$

Ideal direct detection on a coherent state $|\alpha\rangle$ produces outcome "n" (i.e., n "clicks") with probability, $p_n = |\langle n|\alpha\rangle|^2 = |c_n|^2 = \frac{e^{-N} N^n}{n!}$

Poisson detection statistics in a laser pulse is a result of the projection of the quantum state of the laser pulse—a coherent state–on to one of the Fock states

Quantum description of light

- Complete description of an optical field is the **quantum state of a set of mutually-orthogonal modes**
- The most general (pure) state of a mode is an arbitrary superposition of number states of that mode, $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ where $\langle m|n\rangle = \delta_{mn}$, and , $\sum_{n=0}^{\infty} |c_n|^2 = 1$
- Ideal direct detection: $\{|w_k\rangle\} \equiv \{|k\rangle\}$-- number states
- Examples of pure state of a mode:
  - Coherent state $|\alpha\rangle = \sum_{n=0}^{\infty} \left(\frac{e^{-\frac{|\alpha|^2}{2}} \alpha^n}{\sqrt{n!}} \right) |n\rangle \quad , \alpha \in \mathbb{C}$
  - Number (Fock) state, $|n\rangle, n \in \{1, 2, \ldots\}$
  - Cat state: $|\psi_\pm\rangle = \mathcal{N}_\pm \left(|\alpha\rangle \pm |-\alpha\rangle \right)$
    - $\mathcal{N}_\pm = 1/\sqrt{2(1 \pm e^{-2|\alpha|^2})}$ ensures $\langle\psi_+|\psi_+\rangle = 1$, and $\langle\psi_-|\psi_-\rangle = 1$
    - $\langle\psi_+|\psi_-\rangle = 0$ (so, the two cat states can be used to encode a qubit)

Photon detection on cat states

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad |-\alpha\rangle = e^{-\frac{1}{2}|-\alpha|^2} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{\sqrt{n!}} |n\rangle$$

$$|\text{cat}_e\rangle \propto |\alpha\rangle + |-\alpha\rangle$$

$$|\text{cat}_e\rangle \propto 2e^{-\frac{1}{2}|\alpha|^2} \left(\frac{\alpha^0}{\sqrt{0!}}|0\rangle + \frac{\alpha^2}{\sqrt{2!}}|2\rangle + \frac{\alpha^4}{\sqrt{4!}}|4\rangle + \ldots \right)$$

Not Poisson distribution

$$|\text{cat}_o\rangle \propto |\alpha\rangle - |-\alpha\rangle$$

$$|\text{cat}_o\rangle \propto 2e^{-\frac{1}{2}|\alpha|^2} \left(\frac{\alpha^1}{\sqrt{1!}}|1\rangle + \frac{\alpha^3}{\sqrt{3!}}|3\rangle + \frac{\alpha^5}{\sqrt{5!}}|5\rangle + \ldots \right)$$

Entangled states

- Product state of two modes can be written as: $|\psi_1\rangle|\psi_2\rangle$
  with, $|\psi_1\rangle = \sum_{n=0}^{\infty} a_n|n\rangle$ **and** $|\psi_2\rangle = \sum_{n=0}^{\infty} b_n|n\rangle$
  - Tensor product: Each state "lives in its own Hilbert space"
  - Example of product state of 2 modes:
    - Two coherent states, $|\alpha_1\rangle|\alpha_2\rangle$
    - Coherent state and a Fock state, $|\alpha\rangle|k\rangle$
    - Two cat states, $|\psi_+\rangle|\psi_-\rangle$, where $|\psi_\pm\rangle = \mathcal{N}_\pm\left(|\alpha\rangle \pm |-\alpha\rangle\right)$
- Entangled state of two modes cannot be written as $|\psi_1\rangle|\psi_2\rangle$. A two-mode entangled state is:

$$|\psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{n_1,n_2}|n_1\rangle|n_2\rangle \quad , \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} |c_{n_1,n_2}|^2 = 1$$

  - Example (N00N state): $|\psi\rangle = \frac{|n\rangle|0\rangle + |0\rangle|n\rangle}{\sqrt{2}}$

Binary state discrimination

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$$|\psi_1\rangle \ (\text{hypothesis}\, \mathrm{H}_1) \ \text{vs.} \ |\psi_2\rangle \ (\text{hypothesis}\, \mathrm{H}_2)$$

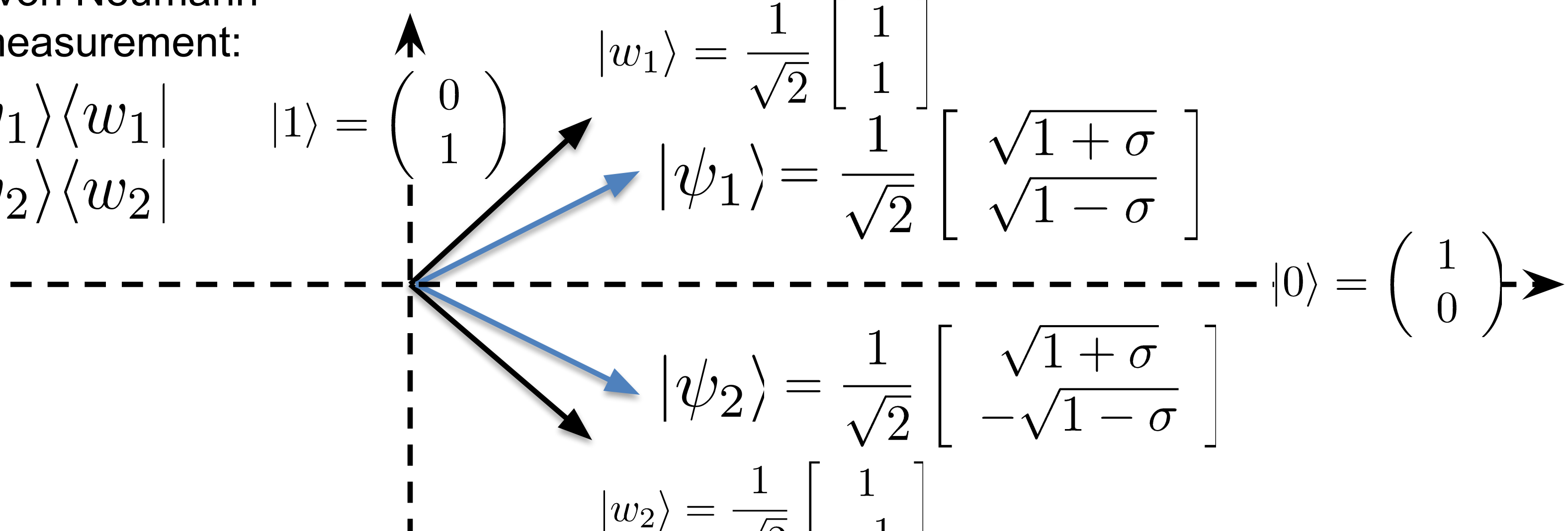
- Assume equal priors: $p_1 = p_2 = \dfrac{1}{2}$

Inner product between the two states

$$\langle \psi_1 | \psi_2 \rangle = \sigma$$

Consider a von Neumann projective measurement:

$$\Pi_1 = |w_1\rangle\langle w_1|$$
$$\Pi_2 = |w_2\rangle\langle w_2|$$



$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|w_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+\sigma} \\ \sqrt{1-\sigma} \end{bmatrix}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1+\sigma} \\ -\sqrt{1-\sigma} \end{bmatrix}$$

$$|w_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P_e = P(\mathrm{H}_1)P(\mathrm{H}_2|\mathrm{H}_1) + P(\mathrm{H}_2)P(\mathrm{H}_1|\mathrm{H}_2)$$
$$= \frac{1}{2}\left|\langle w_2|\psi_1\rangle\right|^2 + \frac{1}{2}\left|\langle w_1|\psi_2\rangle\right|^2$$

This measurement happens to be the one that minimizes the average error probability

Minimum probability of error for binary state discrimination

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$$P_e = \frac{1}{2}\left|\langle w_2|\psi_1\rangle\right|^2 + \frac{1}{2}\left|\langle w_1|\psi_2\rangle\right|^2$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{1+\sigma}\\ \sqrt{1-\sigma}\end{bmatrix} \qquad |w_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{1+\sigma}\\ -\sqrt{1-\sigma}\end{bmatrix} \qquad |w_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$

Problem 3: What is the minimum probability of error in discriminating $|\psi_1\rangle$ and $|\psi_2\rangle$ given their inner product, $\langle\psi_1|\psi_2\rangle = \sigma$? (assume σ is real)

A: $P_e = \sigma$
B: $P_e = [1 - \sqrt{1-\sigma^2}]/2$
C: $P_e = [1-\sigma]/2$
D: I do not know.

MPE decision among M pure states

- M-ary ensemble $\{p_i, |\psi_i\rangle\}\,; i = 1, 2, \ldots, M$
  - Pairwise inner products (Gram matrix), $\sigma_{ij} = \langle\psi_i|\psi_j\rangle$
- Yuen-Kennedy-Lax (YKL) [1975] conditions for MPE
  - Projective measurement, $\{|w_j\rangle\}\,, j = 1, 2, \ldots, M$
  - Relative inner products (aligning measurement vectors in the M-dimensional space spanned by the M pure states):

$$x_{ij} = \langle w_i|\psi_j\rangle$$

  - YKL conditions for minimum average probability of error

(1) $$p_m x_{km} x^*_{mm} = p_k x_{kk} x^*_{mk}$$

(2) $$\sum_{k=1}^{M} x_{kj} x^*_{ki} = \sigma_{ij}$$

(3) $$\sum_{j=1}^{M} x_{jj} |w_j\rangle\langle\psi_j| \geq p_i|\psi_i\rangle\langle\psi_i|,\ \forall i$$

M(M+1)/2 non-linear simultaneous equations: solve for x_{ij}

$$P_{e,\min} = 1 - \sum_{i=1}^{M} p_i |x_{ii}|^2$$

(Check for uniqueness of solution)

Sometimes referred to as "Helstrom bound"

BPSK – minimum probability of error

$$|\alpha\rangle = \sum_{n=0}^{\infty}\left(\frac{e^{-\frac{|\alpha|^2}{2}}\alpha^n}{\sqrt{n!}}\right)|n\rangle$$

- The inner product of two coherent states $|\alpha\rangle$ and $|\beta\rangle$,

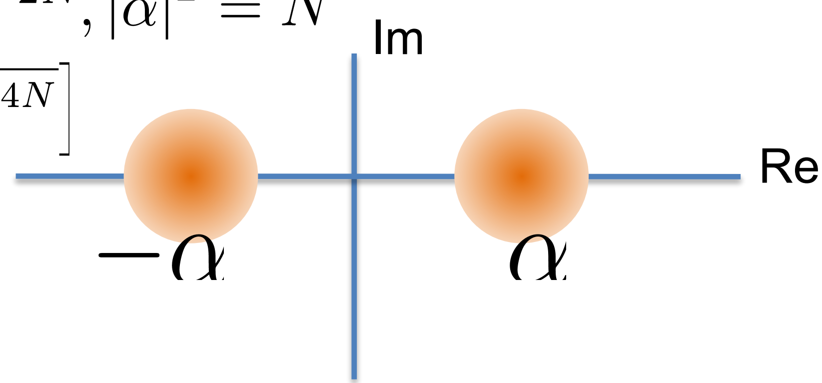
$$\langle\alpha|\beta\rangle = \exp\left[\alpha^*\beta - \frac{1}{2}(|\alpha|^2 + |\beta|^2)\right]$$

- Discriminating BPSK coherent states, $\{|\alpha\rangle, |-\alpha\rangle\}$, $\alpha \in \mathbb{R}$
  - Inner product, $\sigma = \langle\alpha|-\alpha\rangle = e^{-2N}, |\alpha|^2 = N$

$$P_{e.\min} = \frac{1}{2}\left[1 - \sqrt{1 - |\sigma|^2}\right] = \frac{1}{2}\left[1 - \sqrt{1 - e^{-4N}}\right]$$

  - OOK ($|0\rangle, |\alpha\rangle$), $\langle 0|\alpha\rangle = e^{-N/2}$

$$P_{e,\min} = \frac{1}{2}\left[1 - \sqrt{1 - e^{-N}}\right]$$

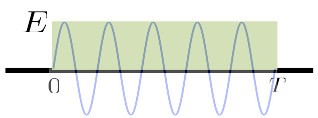


This calculation of minimum error probability using the quantum picture was easy. How do we design a receiver that will achieve this?

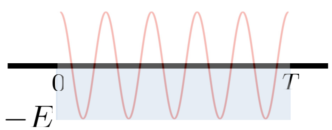
Dolinar's receiver: optimal binary detection

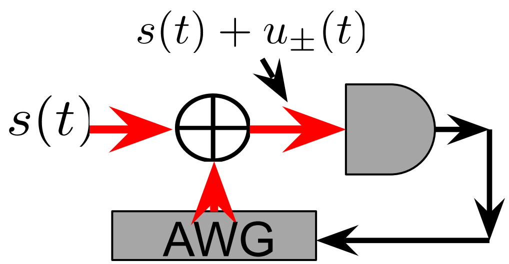
Dolinar, MIT RLE Quarterly Progress Report, 1973

$E^2 T = N$



versus



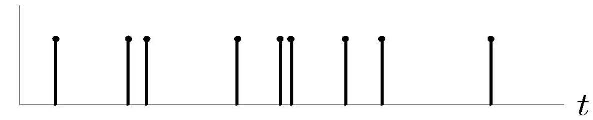


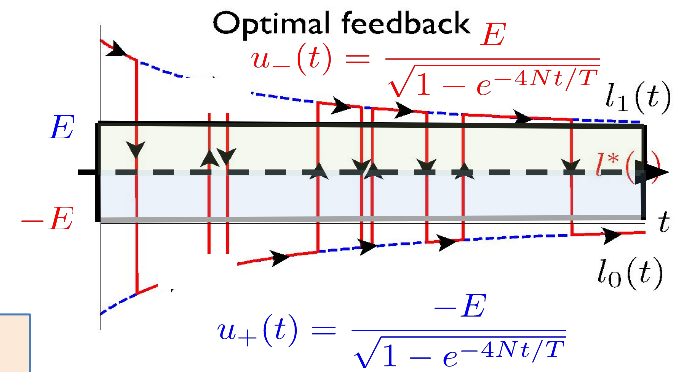
Detector PPP rate:

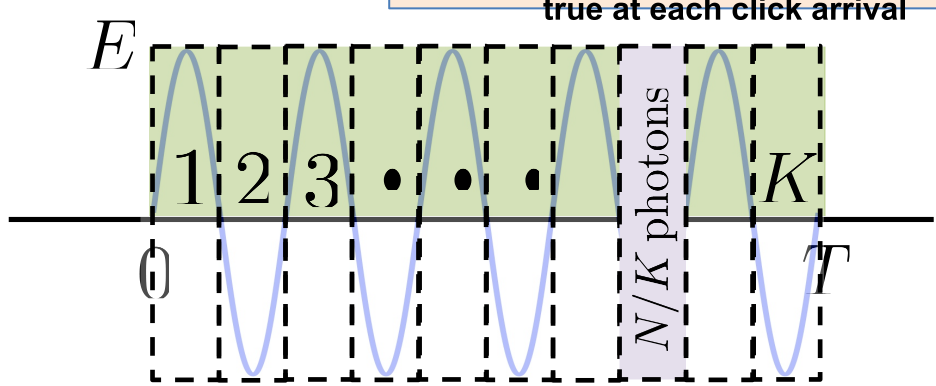
$$\lambda(t) = |s(t) + u_\pm(t)|^2$$

Toggles between applying $u_-(t)$ and $u_+(t)$ at each detector click, and switch receiver's belief of which hypothesis is true at each click arrival

Photon arrivals





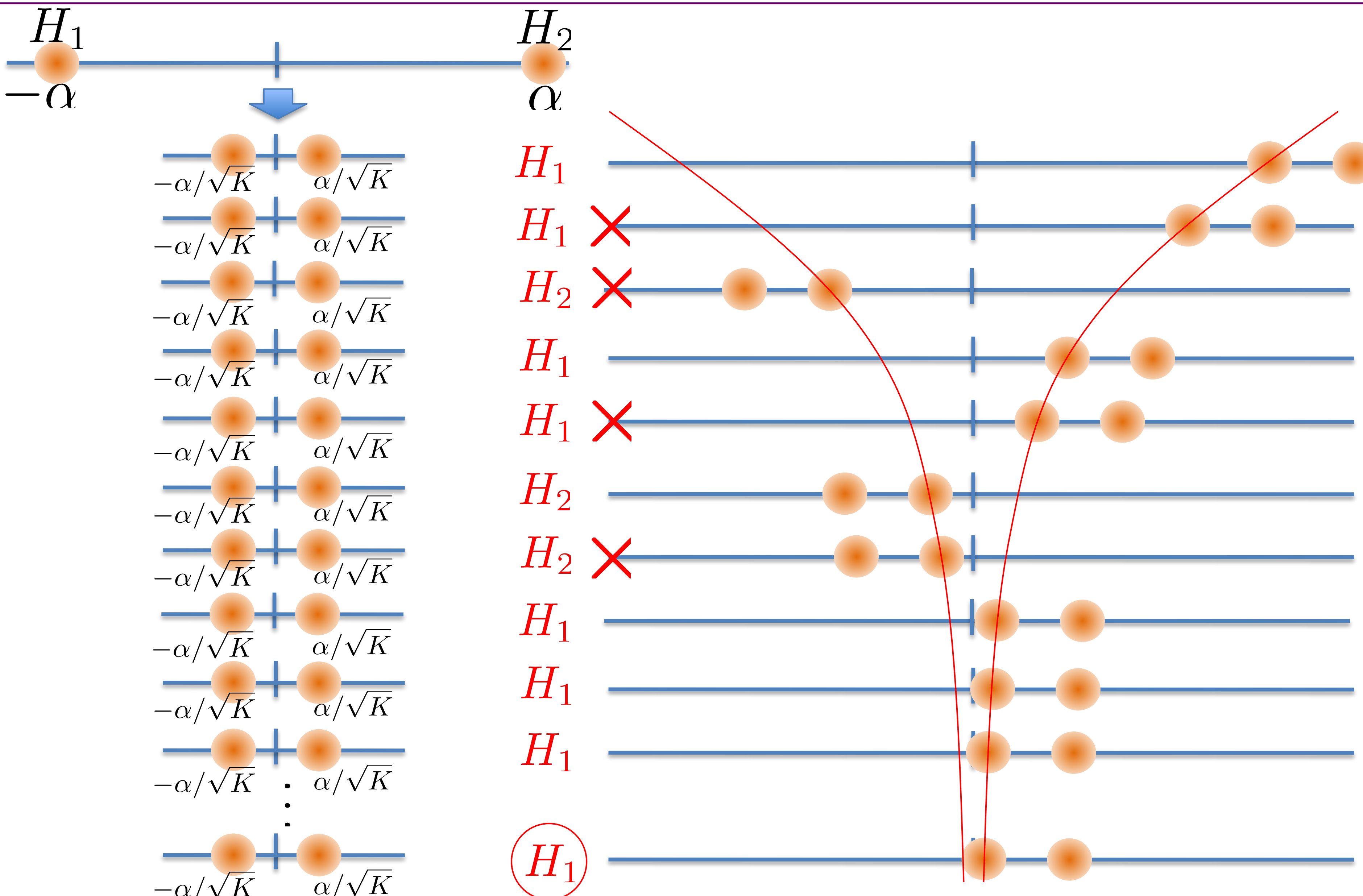


$$P_e = \frac{1}{2}\left[1 - \sqrt{1 - e^{-4N}}\right]$$

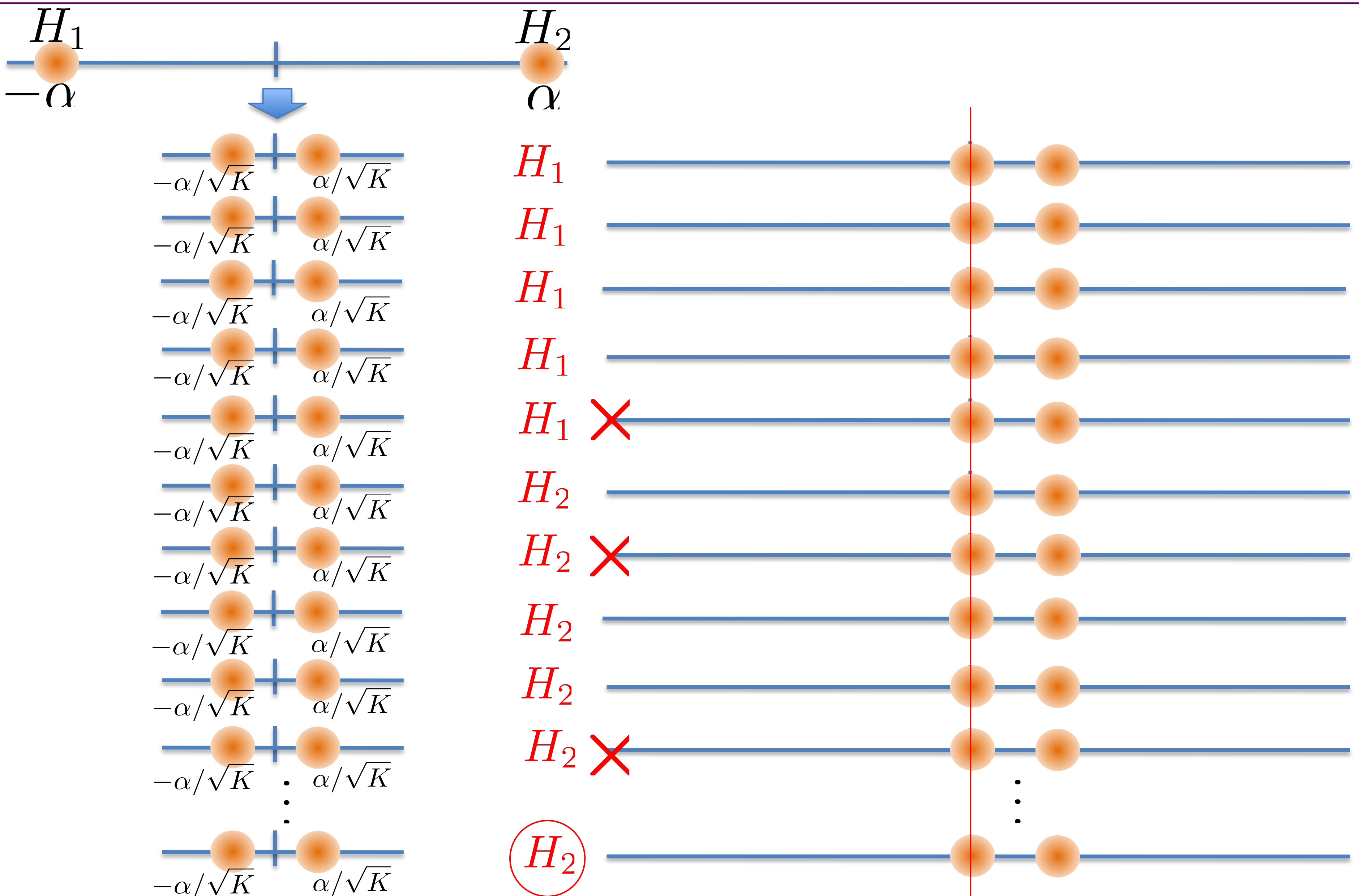
$$|\alpha\rangle \longrightarrow |\alpha/\sqrt{K}\rangle\, |\alpha/\sqrt{K}\rangle\, |\alpha/\sqrt{K}\rangle \ldots |\alpha/\sqrt{K}\rangle$$

$$|-\alpha\rangle \longrightarrow |-\alpha/\sqrt{K}\rangle\, |-\alpha/\sqrt{K}\rangle\, |-\alpha/\sqrt{K}\rangle \ldots |-\alpha/\sqrt{K}\rangle$$

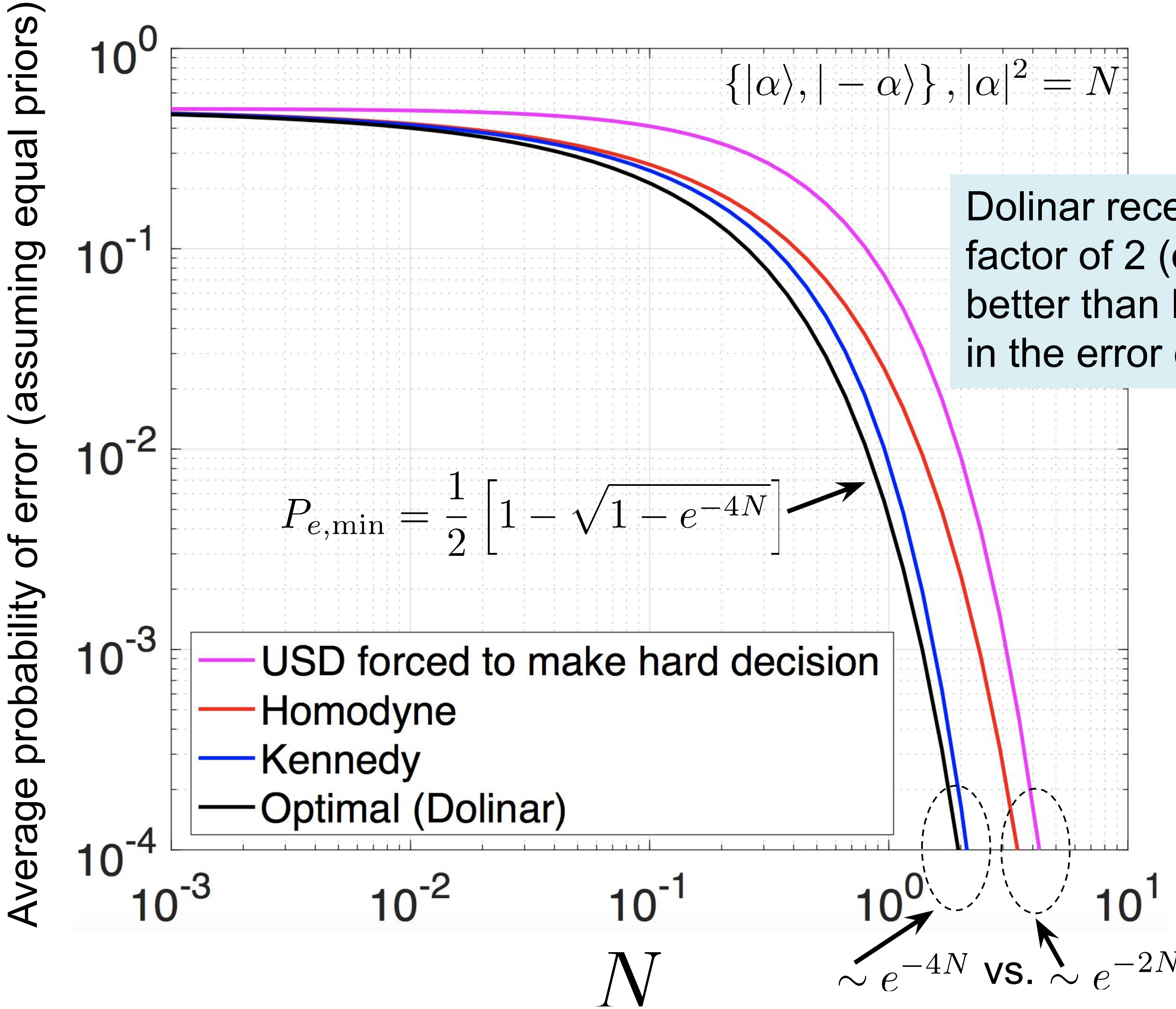
Slicing interpretation of Dolinar's receiver



Slicing interpretation of Kennedy's receiver (exact nulling version)



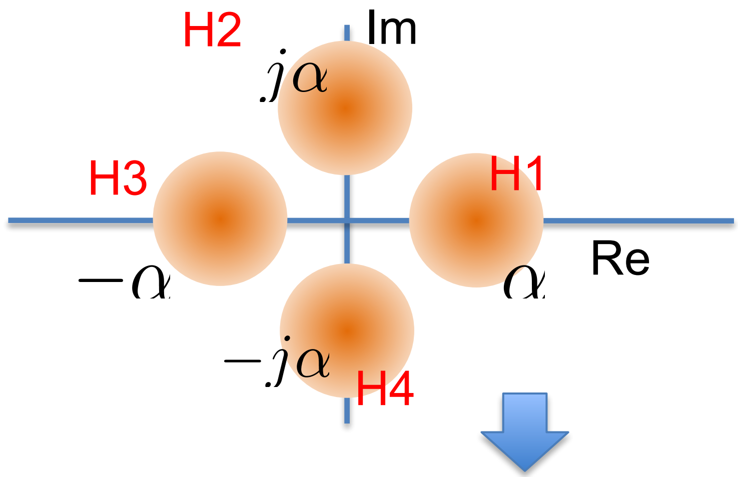
BPSK discrimination performance

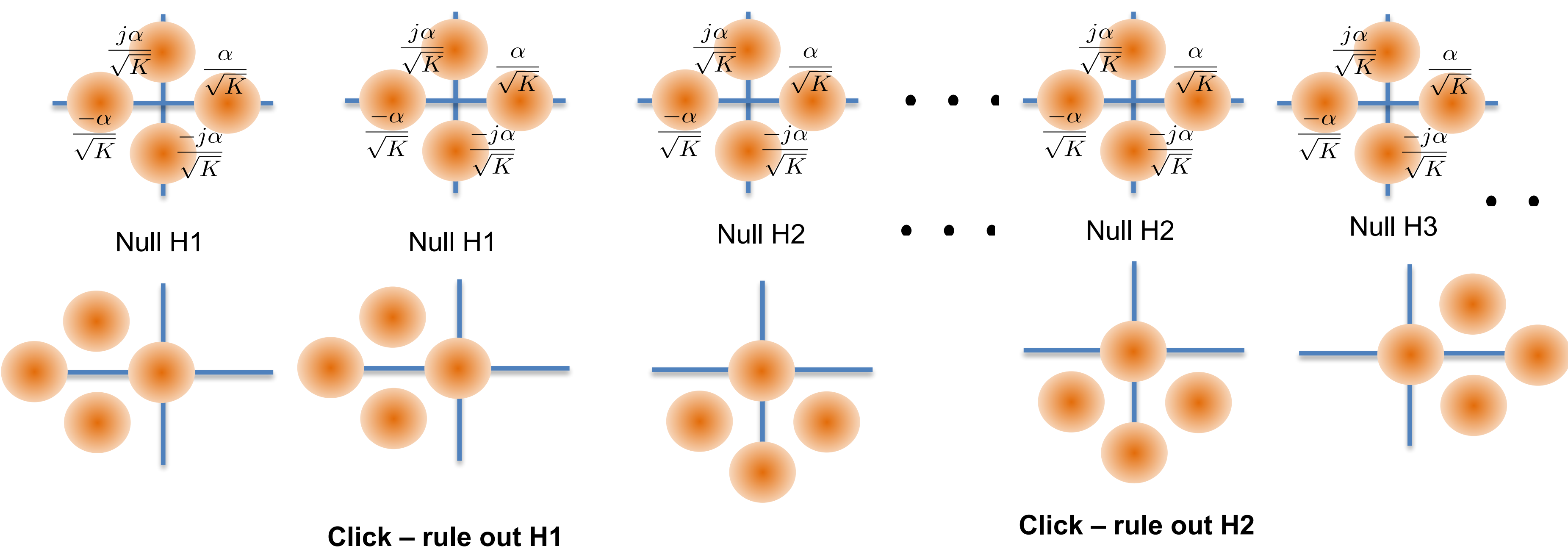


Dolinar receiver is factor of 2 (or, 3 dB) better than homodyne in the error exponent

Bondurant's generalization of exact-nulling Kennedy receiver to Q-ary PSK

Q-ary PSK,
Q=4 shown



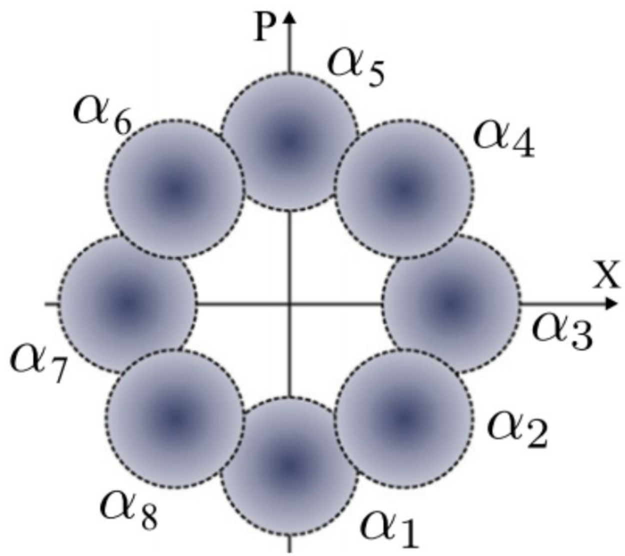


Optimized nulling Bondurant receiver: the equivalent of Dolinar for Q-ary PSK

Center for Quantum Networks NSF-ERC

C. R. Müller and C. Marquardt, New J. Phys. 17 032003 (2015)

They analyze performance of the receiver with all three detector imperfections we discussed



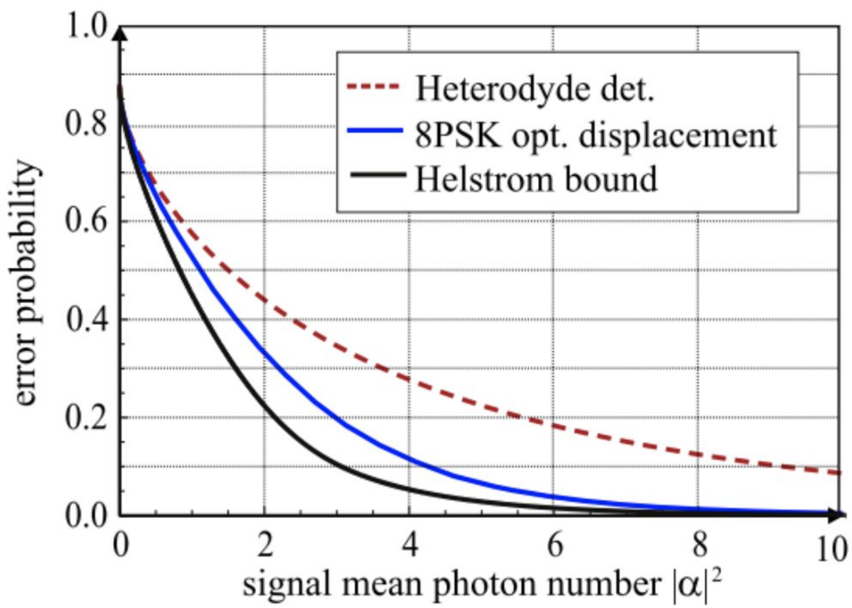


Figure 5. (Left) Illustration of the 8PSK alphabet in phase space. (Right) Error probabilities of the 8PSK optimized displacement receiver (with Bayesian probing) compared to the standard quantum limit (heterodyne detection) and the Helstrom bound. Just like for the QPSK alphabet, the standard quantum limit is outperformed for any signal power.

The "quantum" limit of minimum probability of error cannot be attained by this Dolinar-like receiver strategy. A structured receiver to attain this minimum error probability remains an open problem even for 3 given states of a laser pulse!

Ternary discrimination example

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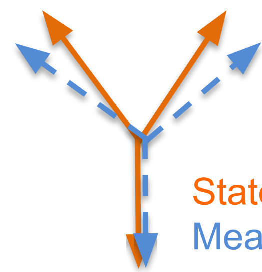
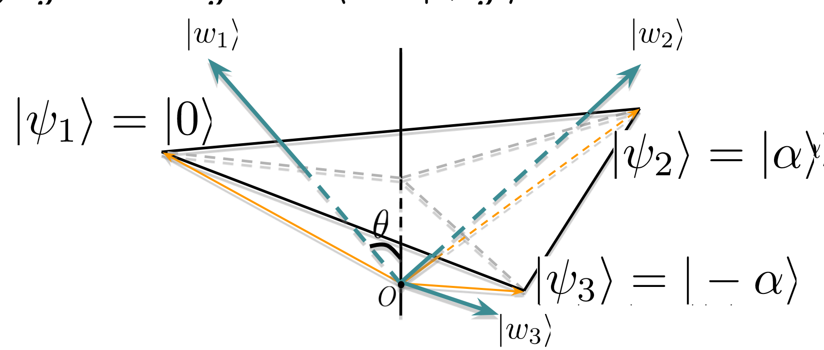
- Consider $|\psi_1\rangle = |0\rangle,\ |\psi_2\rangle = |\alpha\rangle,\ |\psi_3\rangle = |-\alpha\rangle$
  - with α real, equal priors, $N = |\alpha|^2$; what measurement would you use to distinguish these?
- Gram matrix, $((\Gamma))_{ij} = \gamma_{ij} = \langle\psi_i|\psi_j\rangle$



$$\Gamma = \begin{pmatrix} 1 & x & x \\ x & 1 & x^4 \\ x & x^4 & 1 \end{pmatrix}, x = e^{-N/2}$$

- Measurement matrix, $((X))_{ij} = x_{ij} = \langle w_i|\psi_j\rangle$

$$X = \begin{pmatrix} a & d & d \\ b & c & e \\ b & e & c \end{pmatrix}$$



Minimum probability of error

- YKL conditions

$$\sum_{k=1}^{M} x_{kj} x_{ki}^{*} = \gamma_{ij} \quad \begin{cases} a^2 + 2b^2 = 1 \\ d^2 + c^2 + e^2 = 1 \\ ad + b(c+e) = x \\ d^2 + 2ce = x^4 \end{cases}$$

$$p_m x_{km} x_{mm}^{*} = p_k x_{kk} x_{mk}^{*} \quad \{ \; ab = cd$$

  - Solution:

$$a = \left[2xd + (1-x^2)\sqrt{1+x^4-2d^2}\right]/(1+x^4)$$

$$b = \left[x\sqrt{1+x^4-2d^2} - d(1-x^2)\right]/(1+x^4)$$

$$c = \frac{1}{2}\left[\sqrt{1+x^4-2d^2} + \sqrt{1-x^4}\right]$$

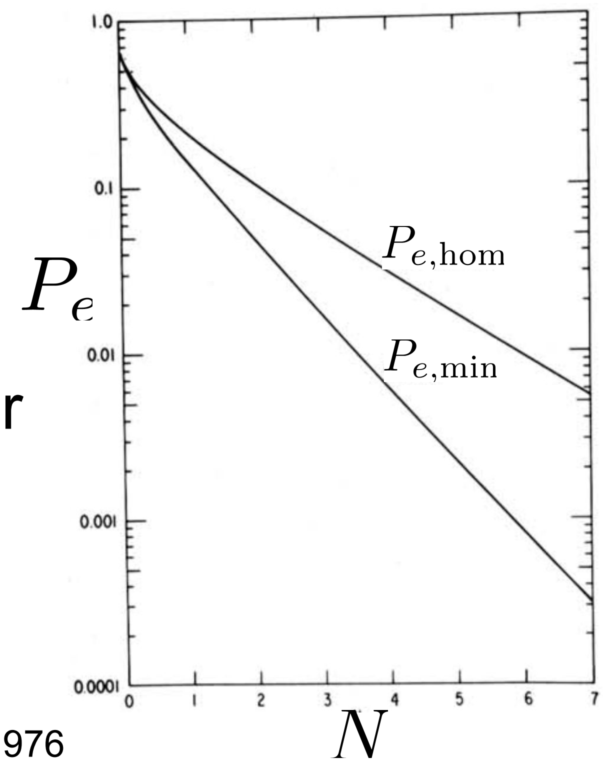
$$e = \frac{1}{2}\left[\sqrt{1+x^4-2d^2} - \sqrt{1-x^4}\right]$$

    - Substitute these into $f(d) = ab - cd$, and solve using Newton's method for the value of d, s.t. f(d) = 0. Evaluate average min probability of error, $P_{e,\min} = 1 - \frac{1}{3}(a^2 + 2c^2)$ and plot as fn. of N

Comparison of MPE with Homodyne

- Evaluate the error probability attained by an ideal homodyne detection receiver: $X \sim \left(x_i, \frac{1}{4}\right), x_i = 0, -\alpha, \alpha$
  - If $|X| < -\alpha/2$, pick $|-\alpha\rangle$, elseif $|X| > \alpha/2$, pick $|\alpha\rangle$, else pick $|0\rangle$
  - Show that, $P_{e,\text{hom}} = \frac{4}{3}\text{erfc}(\sqrt{N})$. Let us plot this.
- Asymptotic limit (N large)
  - For N large, $P_{e,\text{hom}} \sim e^{-N/2}$, whereas $P_{e,\text{min}} \sim e^{-N}$
- Kennedy like receiver

  (sequential-nulling) Kennedy receiver outperforms homodyne, and attains the optimal exponent, exp(-N)



Plot courtesy: Quantum Detection and Estimation Theory, Carl Helstrom, 1976

Another M-ary example: pulse position modulation (PPM)

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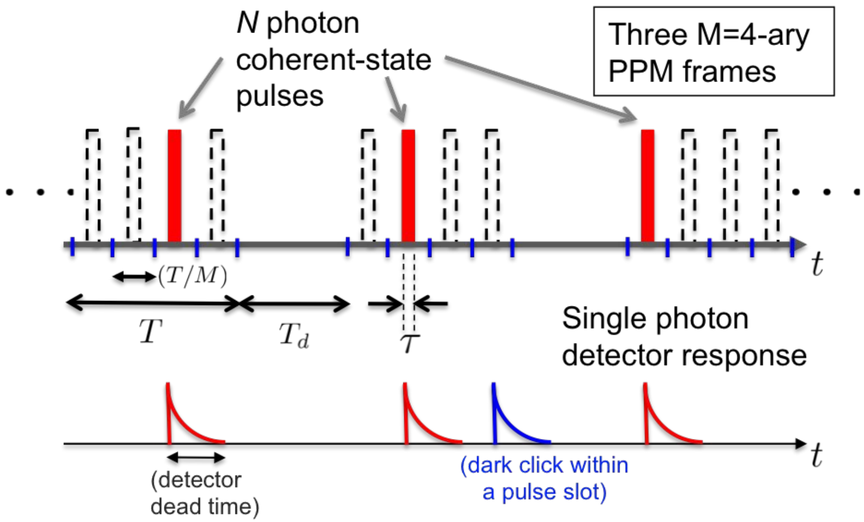
Direct detection

$$|\psi_i\rangle = \underset{1}{|0\rangle} \ldots |0\rangle \underset{i}{|\alpha\rangle} |0\rangle \ldots \underset{M}{|0\rangle} \qquad |\alpha|^2 = N$$



Problem 4: What is the probability of error in discriminating the M-ary PPM codewords achieved by ideal direct (photon) detection on each pulse slot?

A: $P_e = e^{-N}$

B: $P_e = e^{-2N}/M$

C: $P_e = [(M-1)/M]e^{-N}$

D: I do not know.

Another M-ary example: pulse position modulation (PPM)

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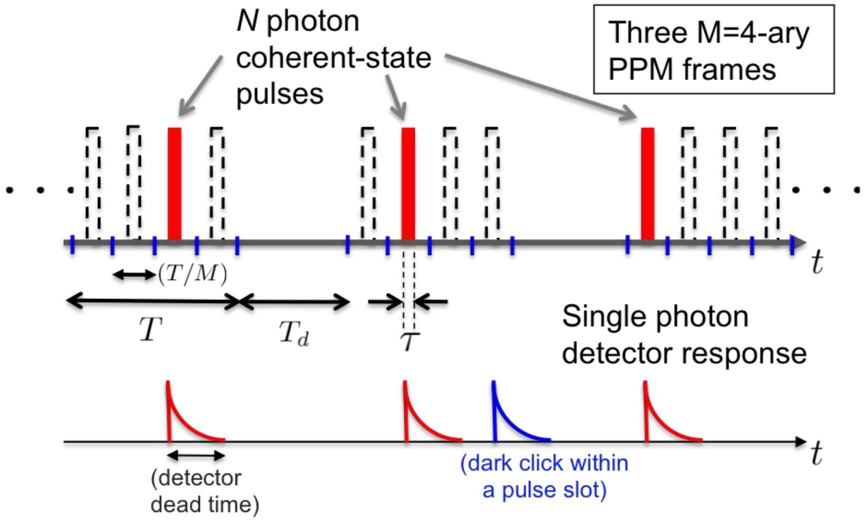
Direct detection

$$|\psi_i\rangle = |0\rangle \ldots |0\rangle|\alpha\rangle|0\rangle \ldots |0\rangle \qquad |\alpha|^2 = N$$

$$1 \quad \ldots \quad i \quad \ldots \quad M$$



$$P_{e,\mathrm{DD}} = \frac{M-1}{M} e^{-N}$$

Quantum MPE limit (YKL)

$$\sigma_{ij} = \langle \psi_i | \psi_j \rangle = |\langle \alpha | 0 \rangle|^2 = e^{-N}, i \neq j$$

$$x_{ij} = b, i \neq j; x_{ii} = a, \forall i$$

(symmetry postulate)

YKL conditions for minimum error probability

$$a^2 + (M-1)b^2 = 1$$

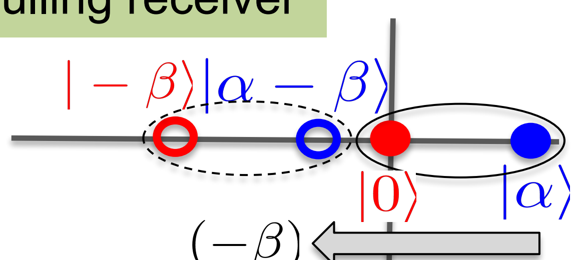
$$2ab + (M-2)b^2 = e^{-N}$$

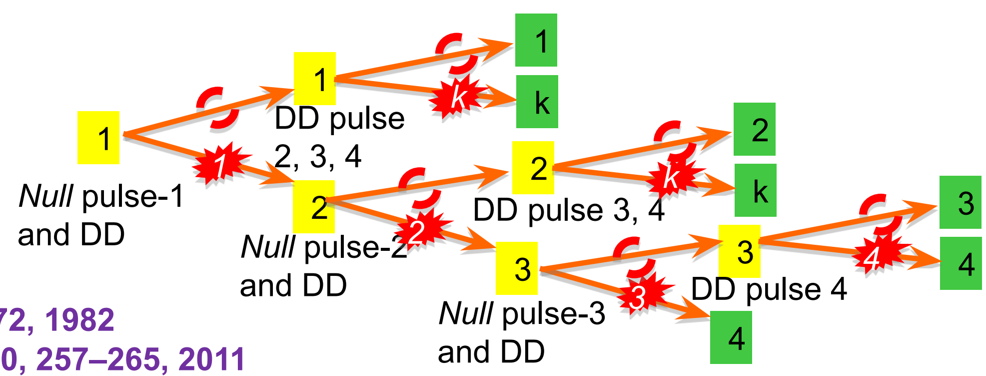
$$P_{e,\min} = 1 - a^2$$

$$= \frac{M-1}{M^2}\left[\sqrt{1+(M-1)e^{-N}} - \sqrt{1-e^{-N}}\right]^2$$

$$\sim e^{-2N}, \; Me^{-N} \ll 1$$

Conditional nulling receiver

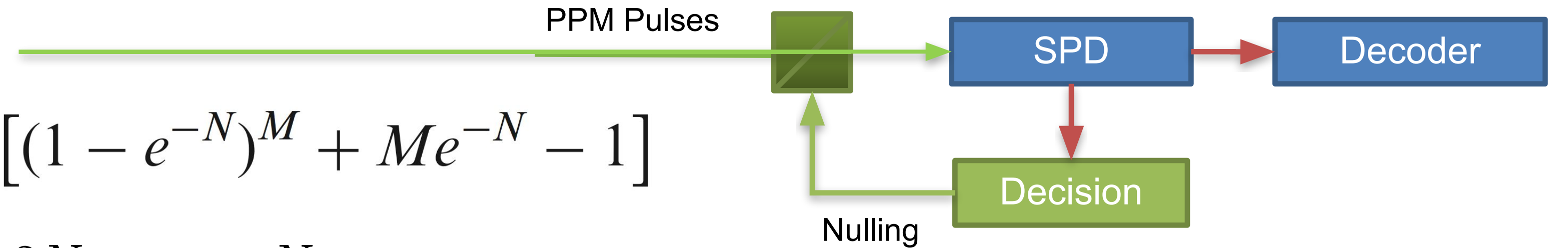




Dolinar, MIT Ph.D. Thesis 1976, TDA Progress Report, 42-72, 1982

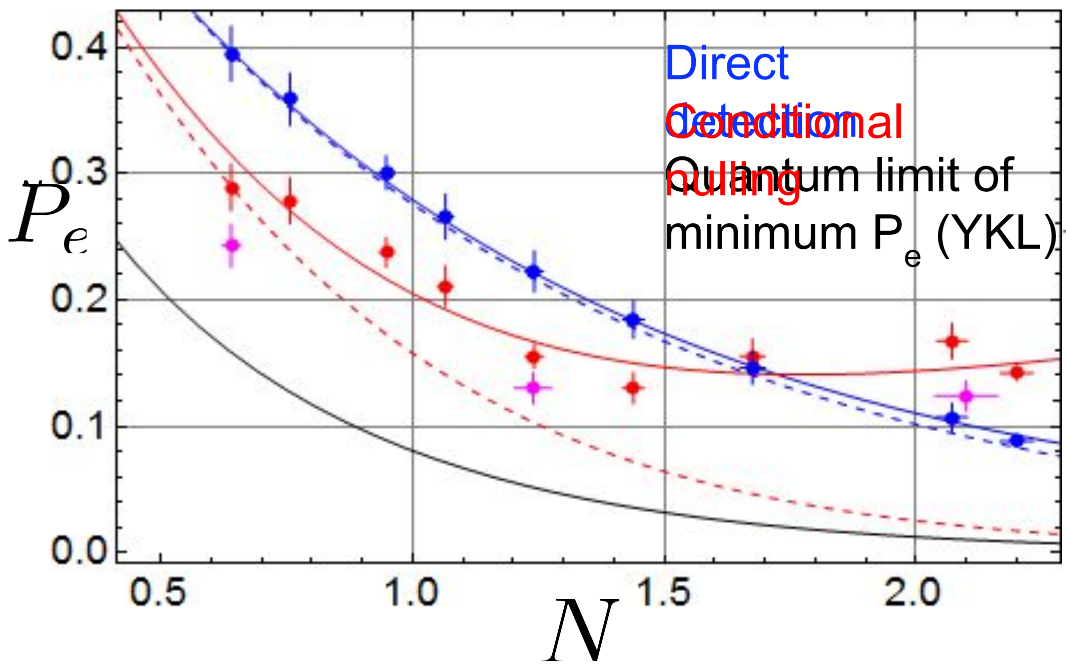
SG, Habif, Takeoka, J. Mod. Optics, Vol. 58, Nos. 3–4, 10–20, 257–265, 2011

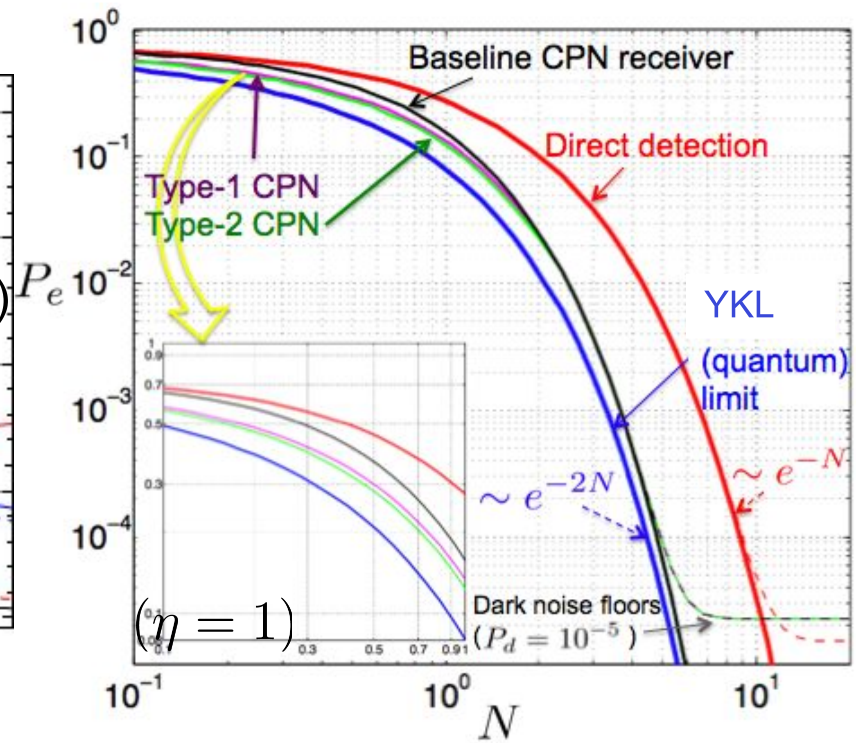
PPM demodulation using the Conditional Pulse Nulling (CPN) receiver



$$P_{e,\mathrm{CPN}} = \frac{1}{M}\left[(1-e^{-N})^M + Me^{-N} - 1\right]$$

$$\sim e^{-2N},\ Me^{-N} \ll 1$$





Chen, Habif, Dutton, Lazarus and SG, Nature Photonics 6, 374-379 (2012)

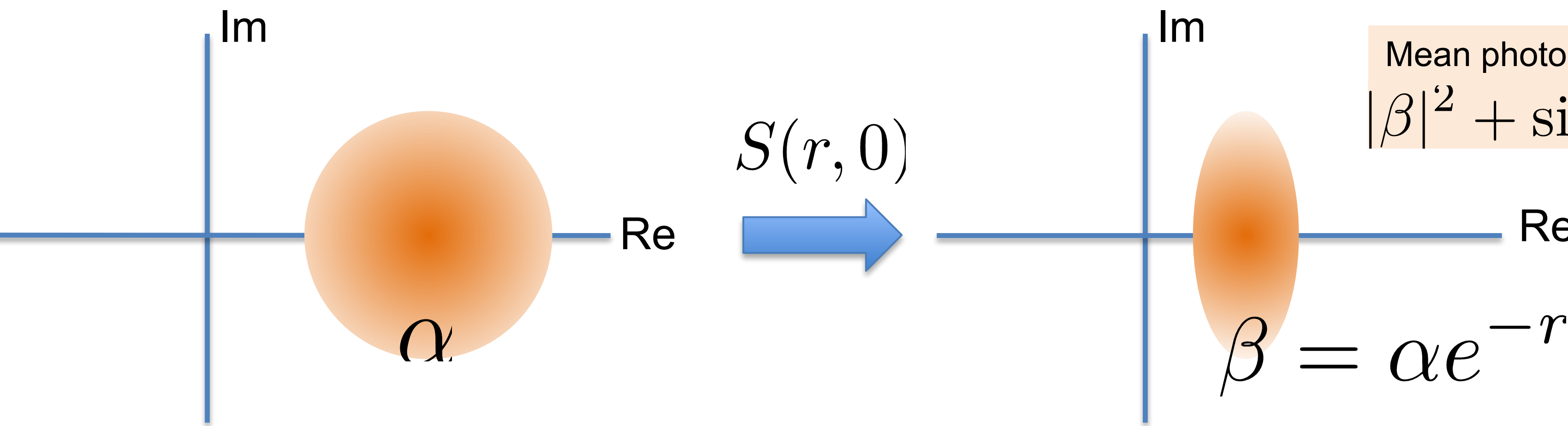
Receiver design that exactly attains the quantum limit is not known.

Universal quantum processing

- We know how to calculate the minimum probability of error for discriminating any M coherent states. Yet, we don't know optimal structured receiver designs for M > 2
- The M = 2 case (Dolinar receiver) was special
- So far, we have been playing with linear optics (circuits of beamsplitters and phases) and direct (photon) detection. These are NOT universal resources
- Adding "squeezing" to our toolbox will make it *universal*
- *We will learn about squeezing in Module 2*

Squeezing

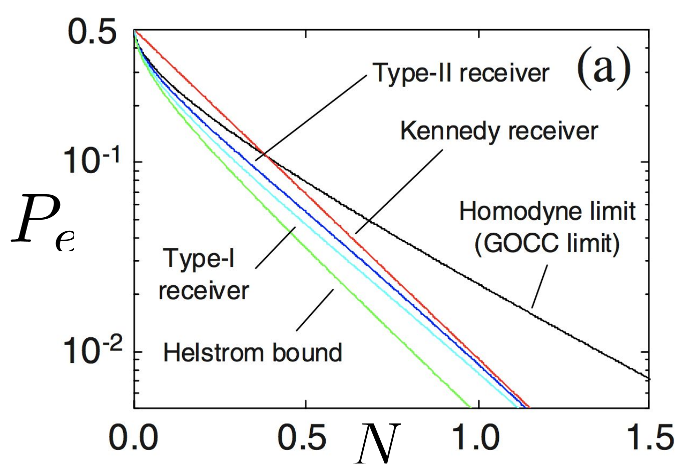
- Squeezing is a unitary transformation, $S(r,\theta)$



Mean photon number $|\beta|^2 + \sinh^2(r)$

- Homodyne detection results in: $X \sim \mathcal{N}\left(\alpha e^{-r}, \frac{1}{4}e^{-2r}\right)$
- Direct detection: $P(k=0) = \cosh(r)e^{-N(1+\tanh(r))}$

Quantum result: cannot be described semi-classically



BPSK discrimination $\{|\alpha\rangle, |-\alpha\rangle\}$: generalization of Kennedy receiver (apply displacement and squeezing before photon detection)

Takeoka and Sasaki, PRA 2008

General design of an optimal receiver

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- Is there a receiver strategy that uses adaptive application of squeezing (not just displacement) on small slices of the coherent state pulses, and photon detection attain arbitrary M-ary MPE state discrimination? [Open problem]
- Instead of an all-optical design, what if we can map each of the BPSK coherent states to a qubit first, i.e.,

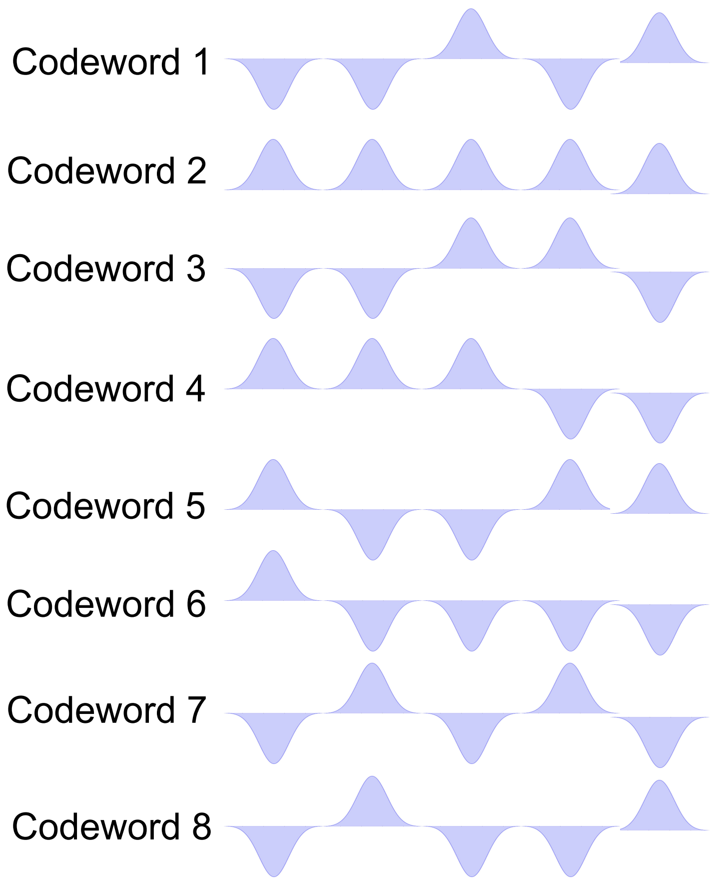
$$|\alpha\rangle \longrightarrow \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{1+\sigma} \\ \sqrt{1-\sigma} \end{bmatrix}$$

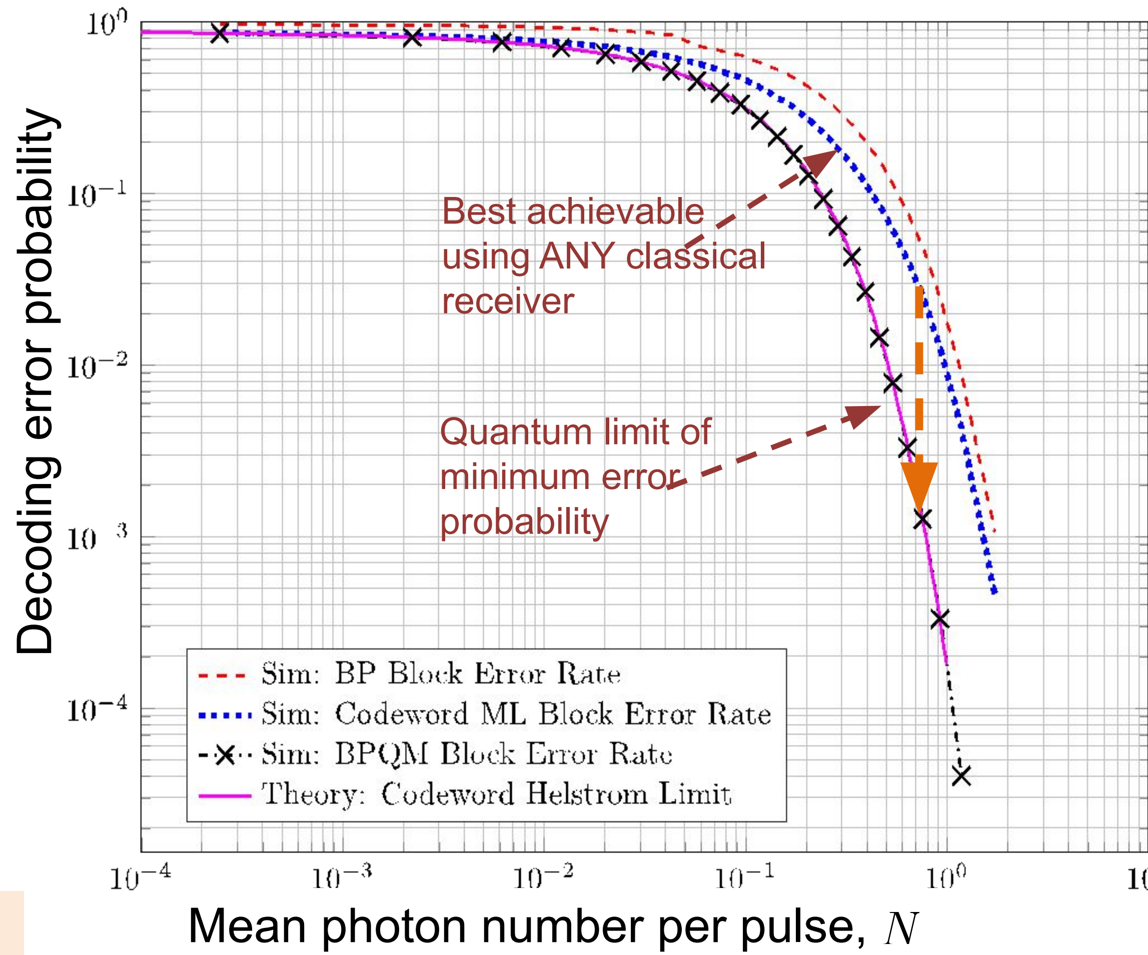
$$|-\alpha\rangle \longrightarrow \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{1+\sigma} \\ -\sqrt{1-\sigma} \end{bmatrix}$$

$$\sigma = \langle -\alpha|\alpha\rangle = e^{-2N},\ N = |\alpha|^2$$

  – Then use *quantum computing* on those qubits?

Quantum limited receiver design





Optical receiver is a "mini quantum computer"

Silva, SG, Dutton, Phys. Rev. A 87, 052320 (2013)
Rengaswamy, Seshadreesan, SG, Pfister, Nature npj Quantum Inf 7, 97 (2021)
Delaney, Seshadreesan, MacCormack, Galda, SG, Narang, Phys. Rev. A 106, 032613 (2022)

Demonstration of quantum advantage

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PHYSICAL REVIEW A **106**, 032613 (2022)

Demonstration of a quantum advantage by a joint detection receiver for optical communication using quantum belief propagation on a trapped-ion device

Conor Delaney,[1] Kaushik P. Seshadreesan,[2,3,*] Ian MacCormack,[1,4,5] Alexey Galda,[6] Saikat Guha,[2] and Prineha Narang[7,†]

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[2]*College of Optical Sciences, The University of Arizona, Tucson, Arizona 85721, USA*
[3]*School of Computing and Information, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA*
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[5]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*
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[7]*School of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138, USA*

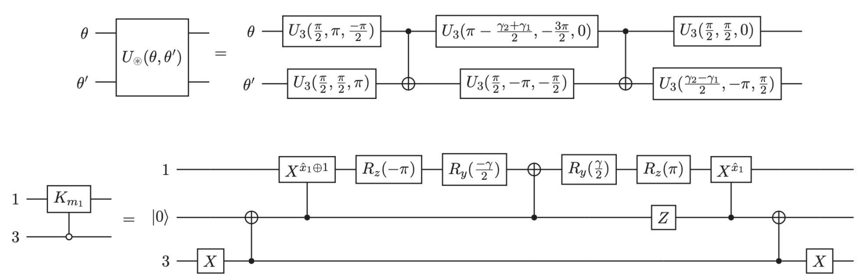
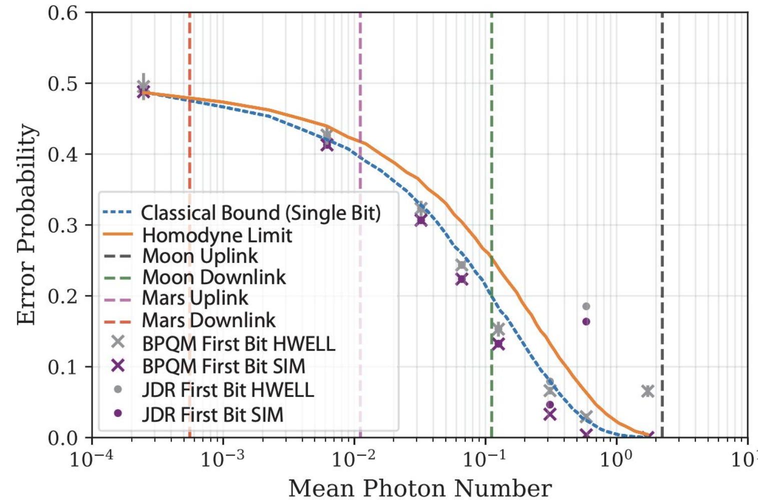
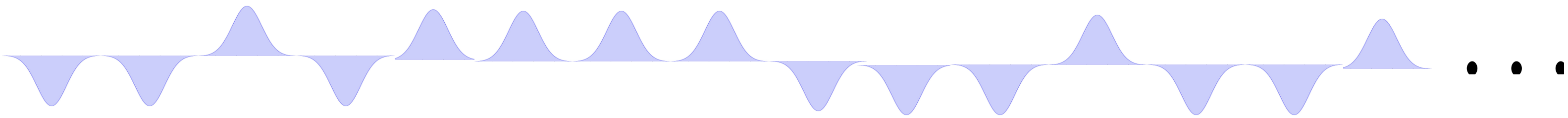


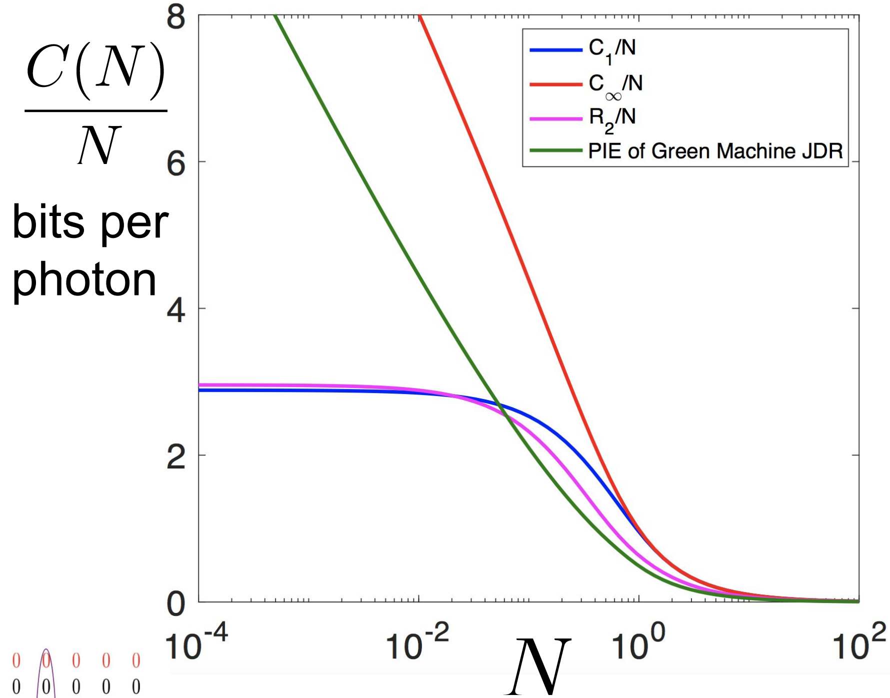
FIG. 5. Both decompositions for BPQM full decoder components. (a) U gate decomposition, where U_3 is the QISKIT rotation gate and γ_1 and γ_2 are defined in Eq. (A9). (b) K_m gate decomposition, utilizing ancilla qubit 3.



Communication capacity

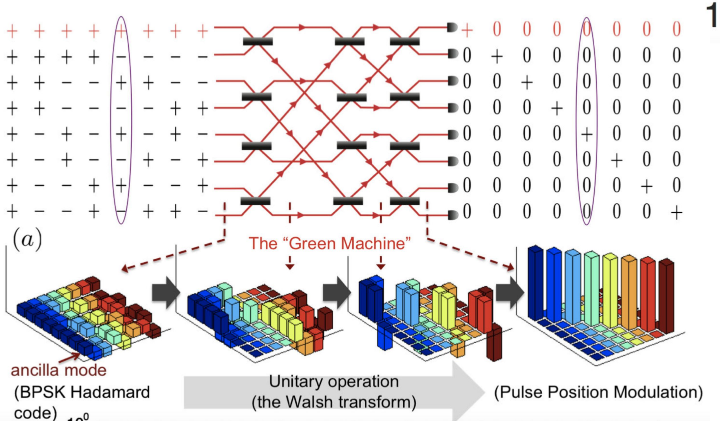


- Mean photon number per received pulse = N
- We can excite each pulse in any coherent state, $|\alpha\rangle, \alpha \in \mathbb{C}$
- How many bits of information C(N) can be faithfully communicated per pulse?



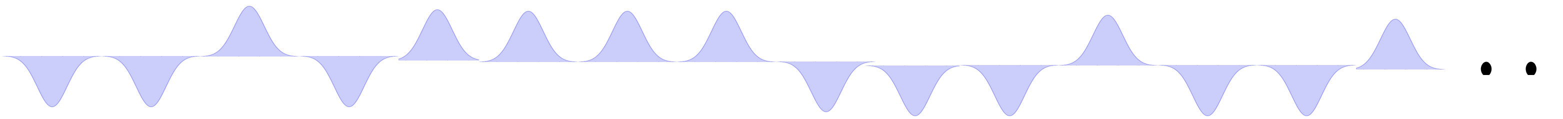
Bridging the remaining gap to the Holevo capacity requires joint-detection receivers that use quantum effects

Chung, SG, Zheng, Phys. Rev. A 96, 012320 (2017)

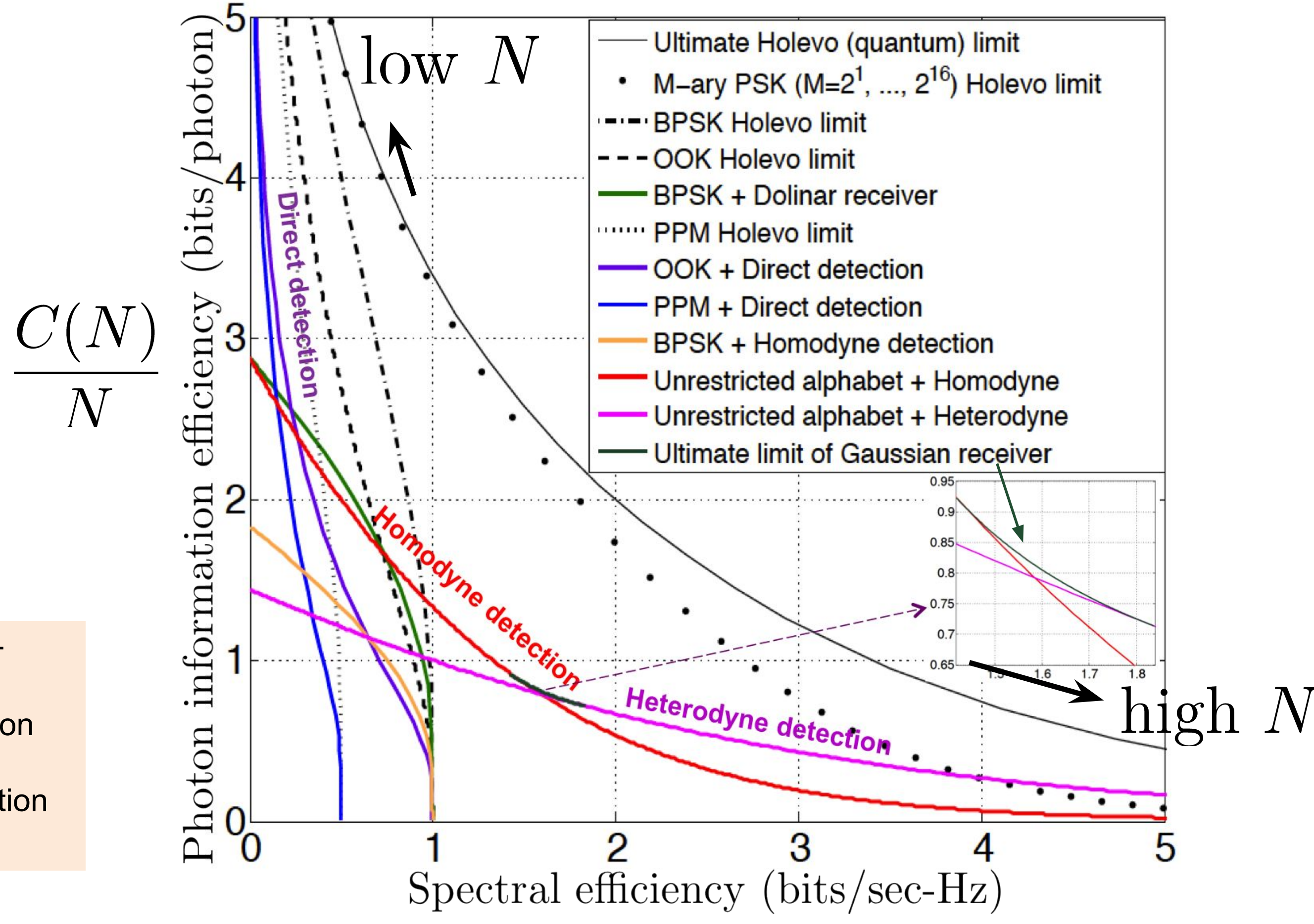


SG, Phys. Rev. Lett. 106, 240502 (2011)

Communication capacity



- Mean photon number per received pulse = N
- We can excite each pulse in any coherent state, $|\alpha\rangle, \alpha \in \mathbb{C}$
- How many bits of information C(N) can be faithfully communicated per pulse?



Shannon capacity: any modulation + receiver combination
Holevo capacity: of a given modulation (optimum joint detection receiver)
Ultimate Holevo capacity: no restriction of modulation or receiver

Takeoka, SG, PRA 89, 042309 (2014)

Module 1: concluding remarks

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- Laser light pulses undergo "wave-like" interference (through beamsplitters), much like ripples of water in a pond
- Laser light field cannot be precisely measured
- Detecting photons on a laser light pulse produces a random number of "clicks" with a Poisson distribution
- Quantum representation of a laser light pulse is a "coherent state" --- this representation helps us quantify "best" receivers (that minimize probability of error, for example), even without knowing how to build such as optimal receiver
- Just using semiclassical tools (interference in a beamsplitter based circuit, and Poisson-noise-limited photon detection), one cannot attain the quantum limit of receiver performance
- Optimal receiver designs require "quantum" processing of laser light – either all-optically using non-classical transformations of light (e.g., using squeezing) or first coupling the laser-light pulses into qubits, followed by processing them in a quantum computer

Break [5 minutes]

Module 2: Quantum information advantage arising from interfering photons

Outline:

1. Introduction to very basic quantum mechanics.
2. Beam-splitters: Classical v Quantum Inputs.
3. Gaussian Multiport interferometers.
4. General Gaussian Transformations and application to Gaussian Boson Sampling.

Quantum Systems

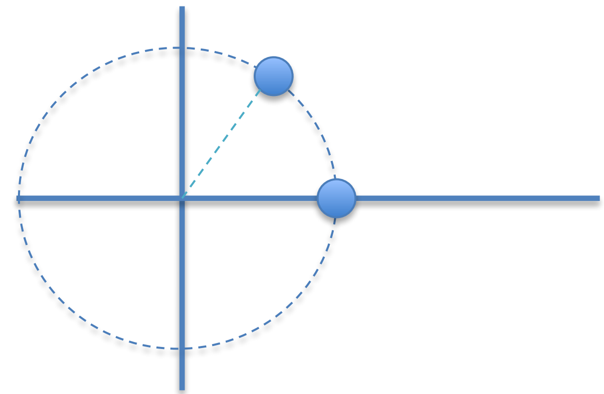
Essentially, Quantum optics is applied linear algebra.

Quantum Systems

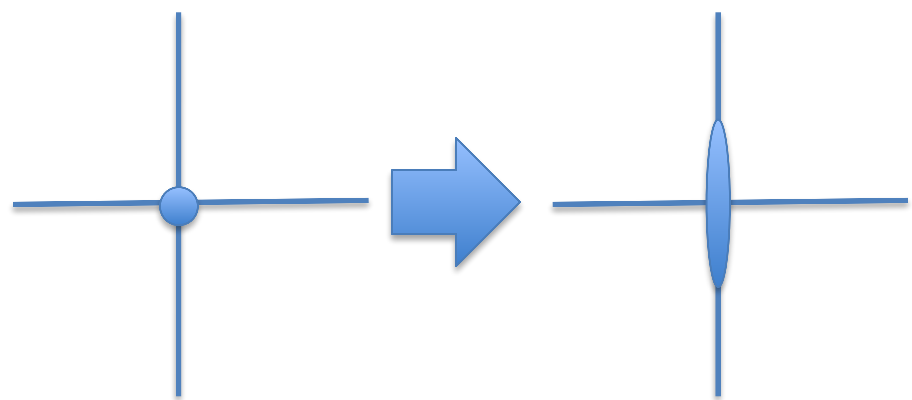
CV System: We encode information in states of light.

Transformation of states

Phase

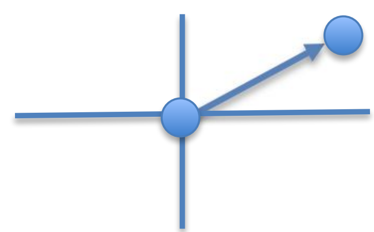


Squeezing



Displacement

$$\hat{U}_\beta = \exp\left(\beta \hat{a}^\dagger - \beta^* \hat{a}\right)$$



Phase space description

$$\chi_W(\zeta^*,\zeta) = \mathrm{tr}(\hat{\rho}\, e^{-\zeta^*\hat{a}+\zeta\hat{a}^\dagger})$$

$$\chi_A(\zeta^*,\zeta) = \mathrm{tr}(\hat{\rho}\, e^{-\zeta^*\hat{a}} e^{\zeta\hat{a}^\dagger})$$

$$\chi_N(\zeta^*,\zeta) = \mathrm{tr}(\hat{\rho}\, e^{\zeta\hat{a}^\dagger} e^{-\zeta^*\hat{a}})$$

Characteristic functions

$$\chi_A(\zeta) = \int Q(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

Q-Function: Always a proper probability density function

$$Q(\alpha) = \frac{1}{\pi^2}\int \chi_A(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

$$\chi_W(\zeta) = \int W(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

May not be a proper probability density function. **Can take negative values.**

$$W(\alpha) = \frac{1}{\pi^2}\int \chi_W(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

$$\chi_N(\zeta) = \int P(\alpha) e^{\zeta\alpha^* - \zeta^*\alpha} d^2\alpha$$

Always a proper probability density function *when it exists.* The states for which a proper P function exists are classical states.

$$P(\alpha) = \frac{1}{\pi^2}\int \chi_N(\zeta) e^{-\zeta\alpha^* + \zeta^*\alpha} d^2\zeta$$

Unitary evolution

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$$\chi_W(\zeta^*,\zeta) = \mathrm{tr}(\hat{\rho}\, e^{-\zeta^*\hat{a}+\zeta\hat{a}^\dagger})$$
$$\chi_A(\zeta^*,\zeta) = \mathrm{tr}(\hat{\rho}\, e^{-\zeta^*\hat{a}}e^{\zeta\hat{a}^\dagger})$$
$$\chi_N(\zeta^*,\zeta) = \mathrm{tr}(\hat{\rho}\, e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}})$$

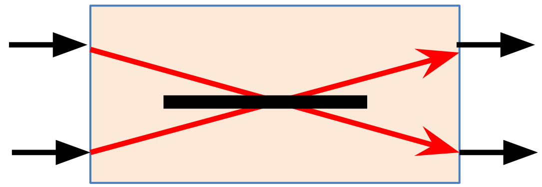
$$\chi_W^U(\zeta^*,\zeta) = \mathrm{tr}\left(\hat{U}\hat{\rho}\hat{U}^\dagger e^{-\zeta^*\hat{a}+\zeta\hat{a}^\dagger}\right) \qquad \chi_W^U(\zeta^*,\zeta) = \mathrm{tr}\left(\hat{\rho}e^{-\zeta^*\hat{U}^\dagger\hat{a}\hat{U}+\zeta\hat{U}^\dagger\hat{a}^\dagger\hat{U}}\right)$$
$$\chi_A^U(\zeta^*,\zeta) = \mathrm{tr}\left(\hat{U}\hat{\rho}\hat{U}^\dagger e^{-\zeta^*\hat{a}}e^{\zeta\hat{a}^\dagger}\right) \qquad \chi_A^U(\zeta^*,\zeta) = \mathrm{tr}\left(\hat{\rho}e^{-\zeta^*\hat{U}^\dagger\hat{a}\hat{U}}e^{\zeta\hat{U}^\dagger\hat{a}^\dagger\hat{U}}\right)$$
$$\chi_N^U(\zeta^*,\zeta) = \mathrm{tr}\left(\hat{U}\hat{\rho}\hat{U}^\dagger e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}\right) \qquad \chi_N^U(\zeta^*,\zeta) = \mathrm{tr}\left(\hat{\rho}e^{\zeta\hat{U}^\dagger\hat{a}^\dagger\hat{U}^\dagger}e^{-\zeta^*\hat{U}^\dagger\hat{a}\hat{U}}\right)$$

Gaussian Unitary Operator: Any unitary operator that maps Gaussian states to Gaussian states.

Two-mode transformation: Beam splitter

Beam splitter $\hat{U}_{\mathrm{BS}} = e^{i\theta\left(\hat{a}\hat{b}^{\dagger} + \hat{a}^{\dagger}\hat{b}\right)}$

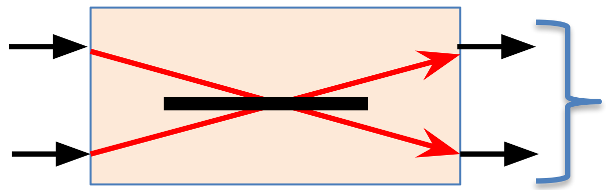
$\hat{\rho}_{\mathrm{in}}$



$\hat{\rho}_{\mathrm{out}}$

$$\hat{a}_{\mathrm{out}} = U^{\dagger}\hat{a}_{\mathrm{in}}\, U = \sqrt{\tau}\hat{a}_{\mathrm{in}} + \sqrt{1-\tau}\hat{b}_{\mathrm{in}}$$

$$\hat{b}_{\mathrm{out}} = U^{\dagger}\hat{b}_{\mathrm{in}}\, U = -\sqrt{1-\tau}\hat{a}_{\mathrm{in}} + \sqrt{\tau}\hat{b}_{\mathrm{in}}$$



Two-mode squeezed vacuum (TMSV)

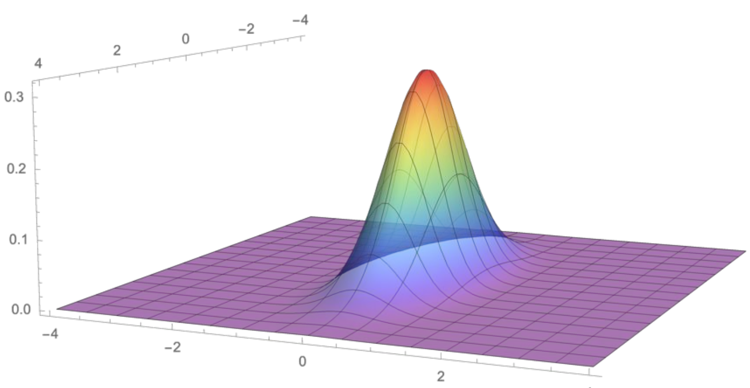
Examples of States Light

Classical and Gaussian

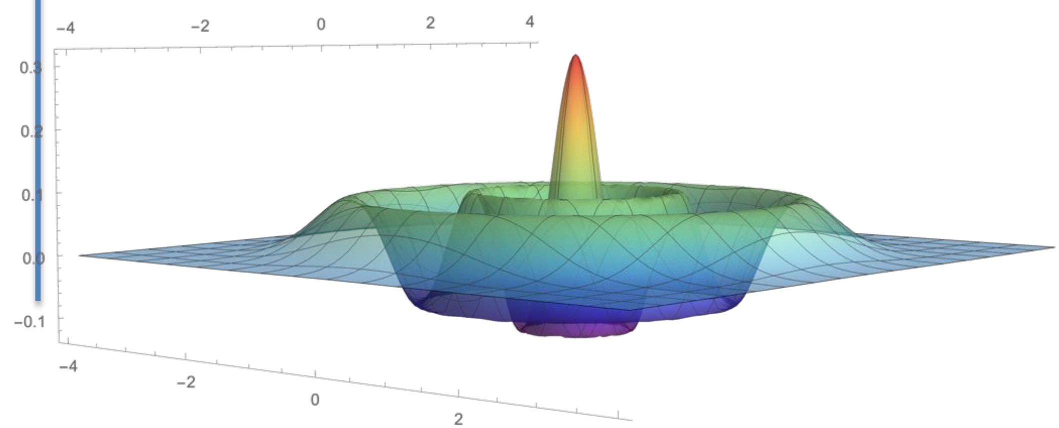
Classical and Non-Gaussian

Mixture of coherent states under
a non-Gaussian PDF:

Quantum and Gaussian



Quantum and Non-Gaussian



Gaussian transformations not universal. Need any one non-Gaussian unitary

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- Phase (1 □ 1)
  –
- Beam splitter (2 □ 2)
  – $$\hat{U}_{\mathrm{BS}} = e^{i\theta\left(\hat{a}\hat{b}^{\dagger}+\hat{a}^{\dagger}\hat{b}\right)}$$
- Squeezing (1 □ 1)
  –
- Displacement (1 □ 1)
  – $$\hat{U}_{\beta} = \exp\left(\beta\hat{a}^{\dagger} - \beta^{*}\hat{a}\right)$$
- Cubic phase (1 □ 1)
  – $$U(\gamma) = e^{i\gamma\hat{q}^{3}}, \; \hat{q} = \frac{\hat{a}+\hat{a}^{\dagger}}{\sqrt{2}}$$

General N-mode passive linear optical transformation

General zero-mean Gaussian unitary.

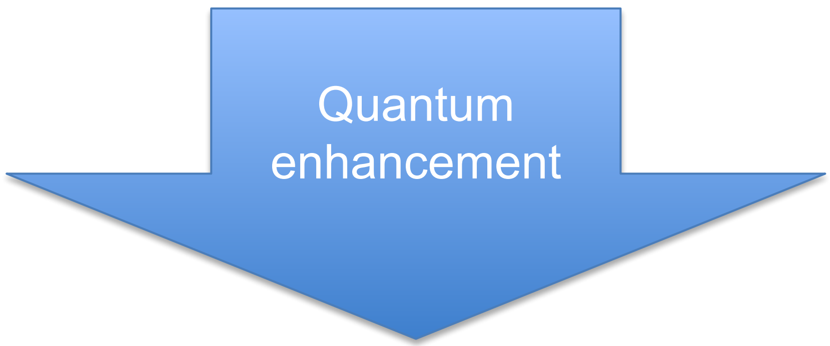
General Gaussian transformation

Seth Lloyd and Samuel L. Braunstein, Phys. Rev. Lett. 82, 1784 (1999)

Julien Niset, Jaromír Fiurášek, and Nicolas J. Cerf, Phys. Rev. Lett. 102, 120501 (2009)

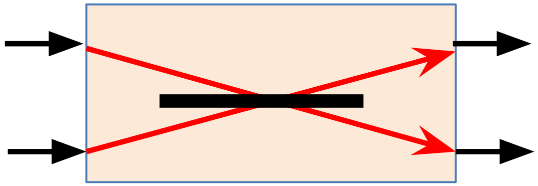
Purely Quantum properties

1. Squeezing (Gaussian Quantum resources).
2. Entanglement.
3. Non-Gaussian states and measurements.



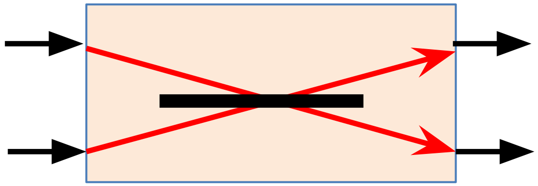
- Sensing
- Communications.
- Universal Quantum Computation

Example on beam splitter calculations



$$\hat{a}_{\text{out}} = U^{\dagger}\hat{a}_{\text{in}}\,U = \sqrt{\tau}\hat{a}_{\text{in}} + \sqrt{1-\tau}\hat{b}_{\text{in}}$$
$$\hat{b}_{\text{out}} = U^{\dagger}\hat{b}_{\text{in}}\,U = -\sqrt{1-\tau}\hat{a}_{\text{in}} + \sqrt{\tau}\hat{b}_{\text{in}}$$

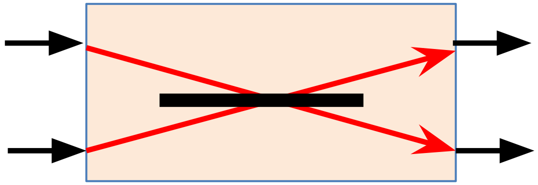
Example on beam splitter calculations (cont.)



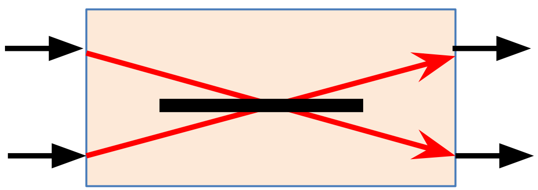
$$\hat{a}_{\text{out}} = U^{\dagger}\hat{a}_{\text{in}}\,U = \sqrt{\tau}\hat{a}_{\text{in}} + \sqrt{1-\tau}\hat{b}_{\text{in}}$$
$$\hat{b}_{\text{out}} = U^{\dagger}\hat{b}_{\text{in}}\,U = -\sqrt{1-\tau}\hat{a}_{\text{in}} + \sqrt{\tau}\hat{b}_{\text{in}}$$

A coherent state input in a beam splitter **cannot** produce entanglement:
The resulting state is a product of two coherent states (while the total mean photon number is conserved).

More examples on beam splitter calculations



$$\hat{a}_{\text{out}} = U^\dagger \hat{a}_{\text{in}} U = \sqrt{\tau}\hat{a}_{\text{in}} + \sqrt{1-\tau}\hat{b}_{\text{in}}$$
$$\hat{b}_{\text{out}} = U^\dagger \hat{b}_{\text{in}} U = -\sqrt{1-\tau}\hat{a}_{\text{in}} + \sqrt{\tau}\hat{b}_{\text{in}}$$



Problems (7 minutes)

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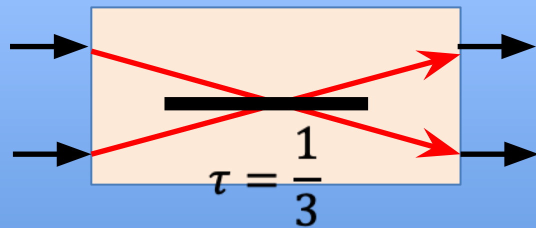
Problem 1: $|1\rangle$

$|1\rangle$



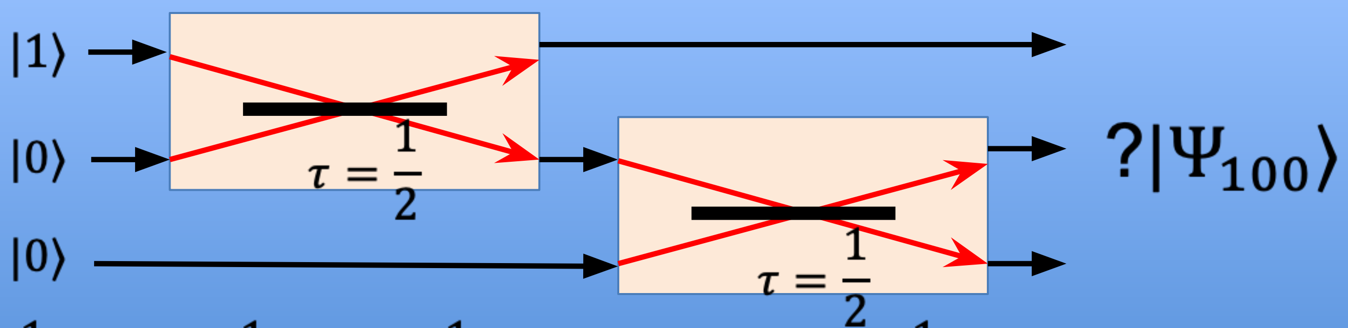
$?|\Phi_{11}\rangle$

A: $|\Phi_{11}\rangle = -\frac{2}{3}|20\rangle - \frac{1}{3}|11\rangle + \frac{2}{3}|02\rangle$ **B:** $|\Phi_{11}\rangle = -\frac{1}{2}|20\rangle + \frac{1}{2}|$

C: $|\Phi_{11}\rangle = \frac{1}{2}|20\rangle + \frac{1}{2}|02\rangle$ **D:** I do not know

Problem

Problem 2:



A: $|\Psi_{100}\rangle = \frac{1}{\sqrt{2}}|100\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|001\rangle$

B: $|\Psi_{100}\rangle = \frac{1}{\sqrt{2}}|110\rangle$

C: $|\Psi_{100}\rangle = \frac{1}{2}|200\rangle + \frac{1}{2}|002\rangle$

D: I do n

Break [5 minutes]

Introducing multi-mode Gaussian interferometers

We have seen that the beam-splitter transforms the annihilation and creation as:

$$\hat{a}_{\text{out}} = U^\dagger \hat{a}_{\text{in}} U = \sqrt{\tau}\hat{a}_{\text{in}} + \sqrt{1-\tau}\hat{b}_{\text{in}}$$
$$\hat{b}_{\text{out}} = U^\dagger \hat{b}_{\text{in}} U = -\sqrt{1-\tau}\hat{a}_{\text{in}} + \sqrt{\tau}\hat{b}_{\text{in}}$$

$$\hat{a}^\dagger_{\text{out}} = \sqrt{\tau}\hat{a}^\dagger_{\text{in}} + \sqrt{1-\tau}\hat{b}^\dagger_{\text{in}}$$
$$\hat{b}^\dagger_{\text{out}} = -\sqrt{1-\tau}\hat{a}^\dagger_{\text{in}} + \sqrt{\tau}\hat{b}^\dagger_{\text{in}}$$

$$\hat{b}_1 = \sqrt{\tau}\hat{a}_1 + \sqrt{1-\tau}\hat{a}_2$$
$$\hat{b}_2 = -\sqrt{1-\tau}\hat{a}_1 + \sqrt{\tau}\hat{a}_2$$

$$\hat{b}^\dagger_1 = \sqrt{\tau}\hat{a}^\dagger_1 + \sqrt{1-\tau}\hat{a}^\dagger_2$$
$$\hat{b}_2 = -\sqrt{1-\tau}\hat{a}^\dagger_1 + \sqrt{\tau}\hat{a}^\dagger_2$$

Multimode Gaussian Unitary evolution

$$\begin{pmatrix} \hat{\vec{b}} \\ \hat{\vec{b}}^{\dagger} \end{pmatrix} = V \begin{pmatrix} \hat{\vec{a}} \\ \hat{\vec{a}}^{\dagger} \end{pmatrix}$$

For example: The **beam splitter** corresponds to: $V = \begin{pmatrix} \sqrt{\tau} & \sqrt{1-\tau} & 0 & 0 \\ -\sqrt{1-\tau} & \sqrt{\tau} & 0 & 0 \\ 0 & 0 & \sqrt{\tau} & \sqrt{1-\tau} \\ 0 & 0 & -\sqrt{1-\tau} & \sqrt{\tau} \end{pmatrix}$

Multimode Gaussian Unitary evolution

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$$\begin{pmatrix} \hat{b} \\ \hat{b}^\dagger \end{pmatrix} = V \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}, \; \begin{pmatrix} \hat{b}^\dagger & \hat{b} \end{pmatrix} \begin{pmatrix} \hat{b} \\ \hat{b}^\dagger \end{pmatrix} = \begin{pmatrix} \hat{a}^\dagger & \hat{a} \end{pmatrix} V^\dagger V \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \Rightarrow 2\hat{b}^\dagger \hat{b} + 1 = \begin{pmatrix} \hat{a}^\dagger & \hat{a} \end{pmatrix} V^\dagger V \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

Problem

Problem 3: Let the single mode squeezer be described by $V = \begin{pmatrix} \cosh r & -\sinh r \\ -\sinh r & \cosh r \end{pmatrix}$

Is V unitary?

A: No.
B: Yes.
C: Depends on the state we transform.
D: I do not know.

Problem

Problem 4: Let the single mode squeezed vacuum state

$$\hat{U}_{\text{squeezer}}|0\rangle = |0;r\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{n!}\left(-\frac{1}{2}\tanh r\right)^{n}|2n\rangle$$

What is the mean photon number of said state?

Reminder: Mean photon number $\bar{n} = \langle 0;r|\hat{n}|0;r\rangle$ and $\hat{n}|n\rangle = n|n\rangle$

A: $\bar{n} = 0$.

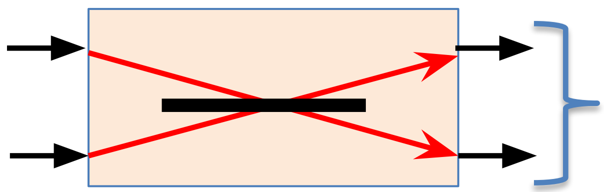
B: $\bar{n} = \sinh^2 r$.

C: $\bar{n} = \frac{1}{2}\sinh^2 r$.

D: I do not know.

Squeezed states: A quantum resource

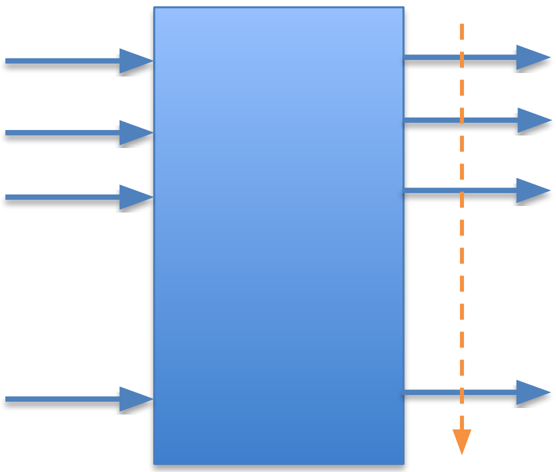
With the math you've learned you can calculate:



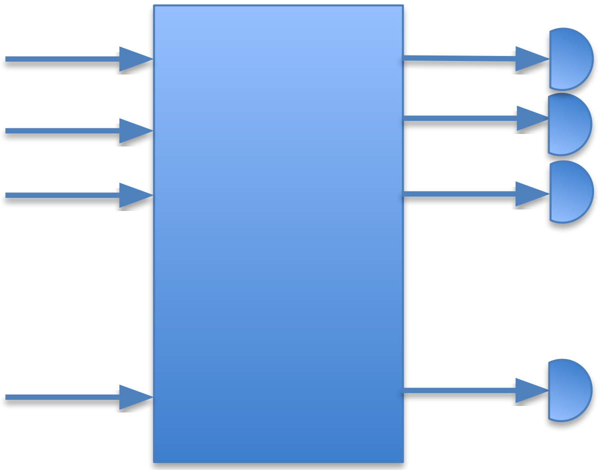
Two-mode squeezed vacuum (TMSV)

The TMSV is the maximally entangled two-mode Gaussian state.

A more general transformation:



The probability of any PNR pattern



$$P_{\vec{n}} = \frac{\left|\mathcal{I}_{n_1 \ldots n_N}\right|^2}{\prod_{i=1}^{N} n_i! 2^{n_i} \cosh r_i}$$

Andrew J. Pizzimenti, Joseph M. Lukens, Hsuan-Hao Lu, Nicholas A. Peters, Saikat Guha, and Christos N. Gagatsos, Phys. Rev. A **104**, 062437 (2021)

$$\mathcal{I}_{\vec{n}} = \int d^N q_\alpha d^N p_\alpha R(\vec{x}_\alpha) \prod_{i=1}^{N} (q_{\alpha_i} + i p_{\alpha_i})^{n_i} = \langle f_1^{n_1} \ldots f_N^{n_N} \rangle, f_i = q_{\alpha_i} + i p_{\alpha_i}$$

$$R(\vec{x}_\alpha) = \frac{1}{(2\pi)^N} e^{-\frac{1}{2}\vec{x}_\alpha^T \mathcal{H} \vec{x}_\alpha}$$

$\mathcal{H}$

The probability of any PNR pattern: Hafnians

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The difficult part in calculating he probability of a photon-number pattern, is the term:

$$\mathcal{I}_{\vec{n}} = \int d^N q_\alpha d^N p_\alpha R(\vec{x}_\alpha) \prod_{i=1}^{N} (q_{\alpha_i} + i p_{\alpha_i})^{n_i} = \langle f_1^{n_1} \dots f_N^{n_N} \rangle, \; f_i = q_{\alpha_i} + i p_{\alpha_i}$$

Hafnian of matrix F:

$$\mathcal{I}_{\vec{n}} = \begin{cases} 0 & \Sigma = \text{odd}, \\ \text{Hf}(F) & \Sigma = \text{even} \end{cases} \qquad \Sigma = \sum_{i=1}^{N} n_i$$

Sampling from a distribution

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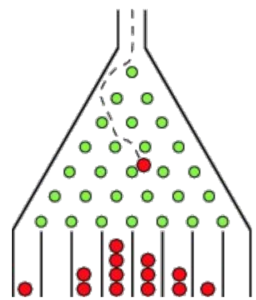
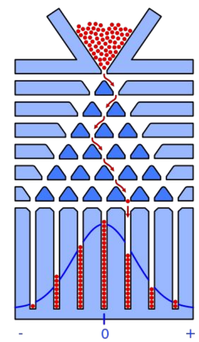
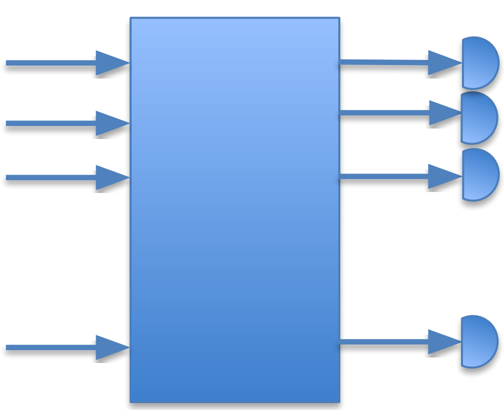




In general, a (classical) computer program that samples from a given distribution would scale exponentially with the size of the requested pattern (e.g. photon-number pattern). Therefore, we program the Hafnian into a quantum-optical circuit, i.e., a *quantum Galton board* (a.k.a. Gaussian Boson Sampler) that performs the sampling job.



Hafnians and perfect matchings

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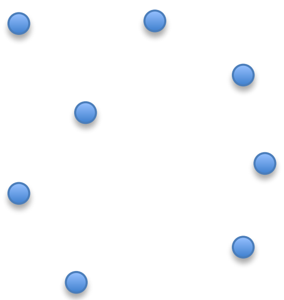
Set of objects



Perfect matching: Connect the objects with a line so that any object is used only once (no more than one line can start/end from any point).

Example:

Perfect matching 1 Perfect matching 2 Perfect matching 3

Hafnians and perfect matchings

How does the previous example relate to the calculation of the Hafnian?

Perfect matching 1 Perfect matching 2 Perfect matching 3

Isserlis' theorem

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$$P_{\vec{n}} = \frac{\left|\mathcal{I}_{n_1 \ldots n_N}\right|^2}{\prod_{i=1}^{N} n_i! 2^{n_i} \cosh r_i} \qquad \mathcal{I}_{\vec{n}} = \int d^N q_\alpha d^N p_\alpha R(\vec{x}_\alpha) \prod_{i=1}^{N} (q_{\alpha_i} + i p_{\alpha_i})^{n_i} = \langle f_1^{n_1} \ldots f_N^{n_N} \rangle, f_i = q_{\alpha_i} + i p_{\alpha_i}$$

$$\langle g_1 g_2 \cdots g_\Sigma \rangle = \sum_{p \in P_\Sigma^2} \prod_{\{i,j\} \in p} \langle g_i g_j \rangle$$

Module 2: Concluding remarks

- Quantum states of light can:
  - Possess counter-intuitive properties
  - Give quantum advantage

- Necessary tools to manipulate all the different protocols:
  - Linear algebra.
  - Building intuition.

Quantum supremacy (advantage)

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- Google, US Nature volume 574, pages 505–510 (2019)
  - IQP: a random circuit based sampler [53-qubit circuit of depth 20, with 430 two-qubit and 1,113 single-qubit gates. Classical simulation estimate ~ 10,000 years based on a Schrödinger-Feynman simulation that trades off space for time, whereas Sycamore processor takes about 200 seconds
- Xanadu, Canada Nature volume 606, pages 75–81 (2022)
  - Gaussian Boson Sampling: 216 squeezed modes entangled with three-dimensional connectivity; it would take ~ 9,000 years for the best available algorithms and supercomputers to produce, using exact methods, a single sample from the programmed distribution, whereas Borealis requires only 36 μs
- USTC, China Science, Vol 370, Issue 6523 pp. 1460-1463, 3 Dec 2020
  - Gaussian Boson Sampling: 50 squeezed modes in a 100-mode interferometer; measured a sampling rate that is about 10^{14}-fold faster than using state-of-the-art classical simulation strategies and supercomputers (2.5 billion years as opposed to 200 s)

What are random samplers good for?

- Quantum generative adversarial networks (arXiv:1804.08641)
- Neural networks and other machine learning algorithms powered by continuous-variable quantum states and processes (arXiv:1806.06871, arXiv:1912.08278, arXiv:2001.03622)
- Graph isomorphism and graph similarity testing (arXiv:1810.10644)
- Molecular vibronic spectra for quantum chemistry calculations (arXiv:1912.07634)
- Quantum adiabatic optimization algorithm or QAOA (arXiv:1902.00409)
- Producing samples from hard-to-generate stochastic point processes (arXiv:1906.11972)
- Multi-variate optimization (arXiv:1909.02108)

Random samplers (Boson sampling, IQP, GBS, etc.) are not universal quantum computers. But they are believed to be strictly more powerful than classical computers

Conclusion

- The building blocks we covered in Modules 1 and 2 (linear optics, coherent states, multi-photon states, squeezed states, photon detection, homodyne detection) in principle suffice to design “quantum-optimal” transmitters, processors, computers, receivers, for all applications in photonics-based information processing
- Yet, we don’t know how to “put together” these building blocks to attain best performance in most applications of photonic information processing!

Course Evaluation Survey

We value your feedback on all aspects of this short course. Please go to the link provided in the Zoom Chat or in the email you will soon receive to give your opinions of what worked and what could be improved.

CQN Winter School on Quantum Networks

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